Information Design for Congestion Games with Unknown Demand

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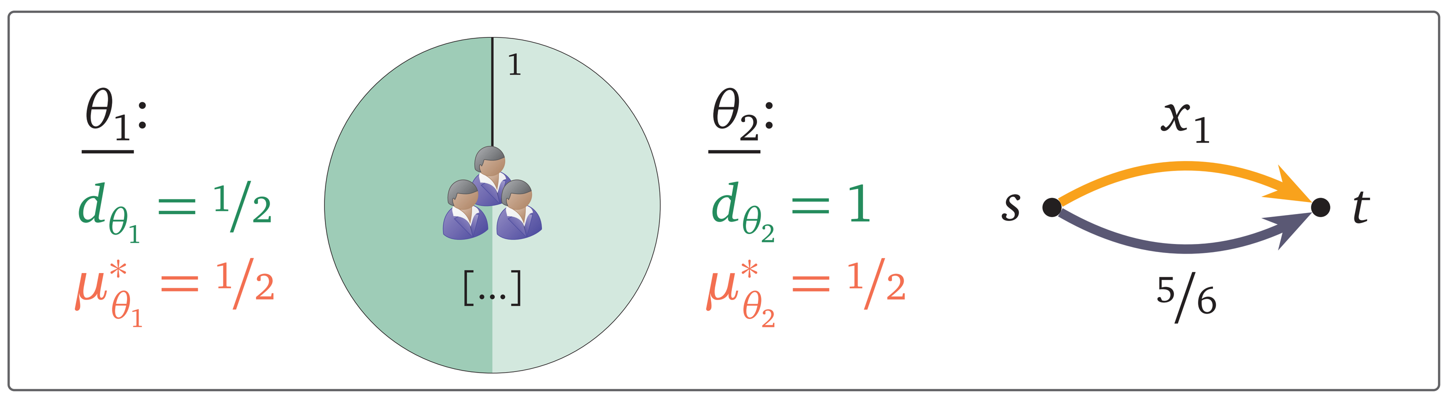
Model & Concepts



- *Non-atomic Congestion Games:*
  Continuum of players with available demand $D = 1$ selfishly travels from $s \in V$ to $t \in V$ in directed graph $G = (V, E)$ with affine edge costs
  $$c_e(x_e) = a_e \cdot x_e + b_e, \quad a_e \in \mathbb{R}_{>0},\ b_e \in \mathbb{R}_{\geq 0}$$

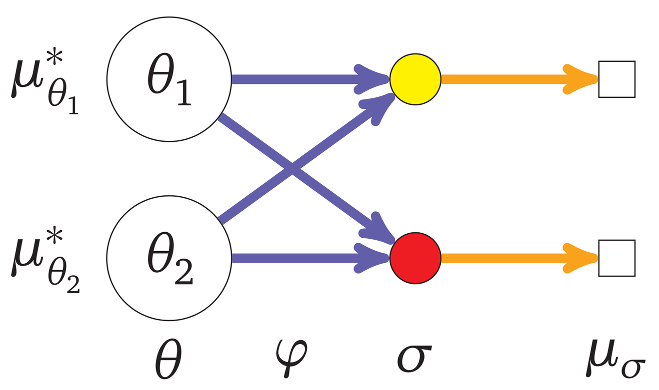
- *Incomplete Information:*
  Actual demand of players is determined by state of nature θ drawn from $\Theta = \{\theta_1, \dots, \theta_\ell\}$ based on probability distribution $\mu^* \in \Delta(\Theta)$.

  Each θ entails an actual demand $d_\theta \leq 1$, i.e., each player is active independently with probability d_θ.

- *Information Design:*
  Principal $\mathscr{P}$ knows actual θ exclusively (players only see μ^*).

  $\mathscr{P}$ sends a signal $\sigma \in \Sigma$ to all active players according to signaling scheme $\varphi = (\varphi_{\theta,\sigma})_{\theta\in\Theta,\sigma\in\Sigma}$ where $\varphi_{\theta,\sigma} \geq 0$ is the joint probability that state θ is realized and signal σ is sent.

  Upon receiving σ (knowing μ^* and φ), active players infer conditional belief $\mu_\sigma \in \Delta(\Theta)$ about the actual demand.



$$\varphi = \begin{pmatrix} \varphi_{\theta_1,\color{red}\bullet} & \varphi_{\theta_1,\color{gold}\bullet} \\ \varphi_{\theta_2,\color{red}\bullet} & \varphi_{\theta_2,\color{gold}\bullet} \end{pmatrix}$$

$$\varphi_\sigma = \sum_{\theta\in\Theta} \varphi_{\theta,\sigma}$$

Every signaling scheme φ can be seen as a convex decomposition of μ^* into induced beliefs μ_σ since $\mu^*_\theta = \sum_{\sigma\in\Sigma} \varphi_{\theta,\sigma} = \sum_{\sigma\in\Sigma} \varphi_\sigma \cdot \mu_{\theta,\sigma}$.

- Each active player depends their choice of an s-t-path with minimum private cost on expected edge cost conditioned on μ_σ:
  $$c_e(x_e \mid \mu_\sigma) = \sum_{\theta\in\Theta} \mu_{\theta,\sigma} \cdot d_\theta \cdot c_e(d_\theta \cdot x_e)$$

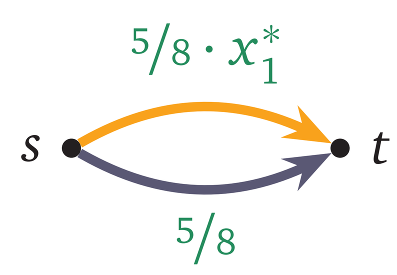
- They reach a Wardrop equilibrium $x^*(\mu_\sigma)$ (WE) with support A_σ of used edges and total expected cost $C(\mu_\sigma) = \sum_{e\in E} x^*_e(\mu_\sigma) \cdot c_e(x^*_e(\mu_\sigma) \mid \mu_\sigma)$.

- $\mathscr{P}$ is benevolent and aims to compute optimal scheme φ^* that minimizes the total expected cost of the resulting WE, i.e.,
  $$C(\varphi) = \sum_{\sigma\in\Sigma} \varphi_\sigma \cdot C(\mu_\sigma)$$

Consider $\Sigma = \{a, b\}$. Let $\mu := \mu_{\theta_2}$.

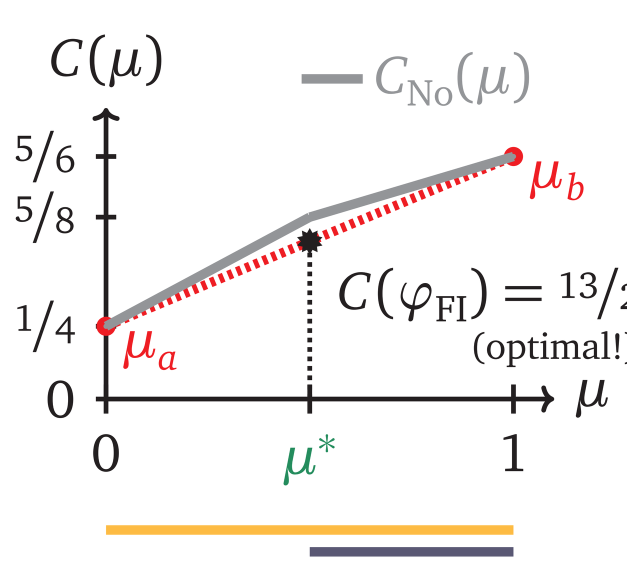
No Signal (No):

$$\varphi_{\text{No}} = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$



$C(\varphi_{\text{No}}) = 5/8$

Full Information (FI):



Results

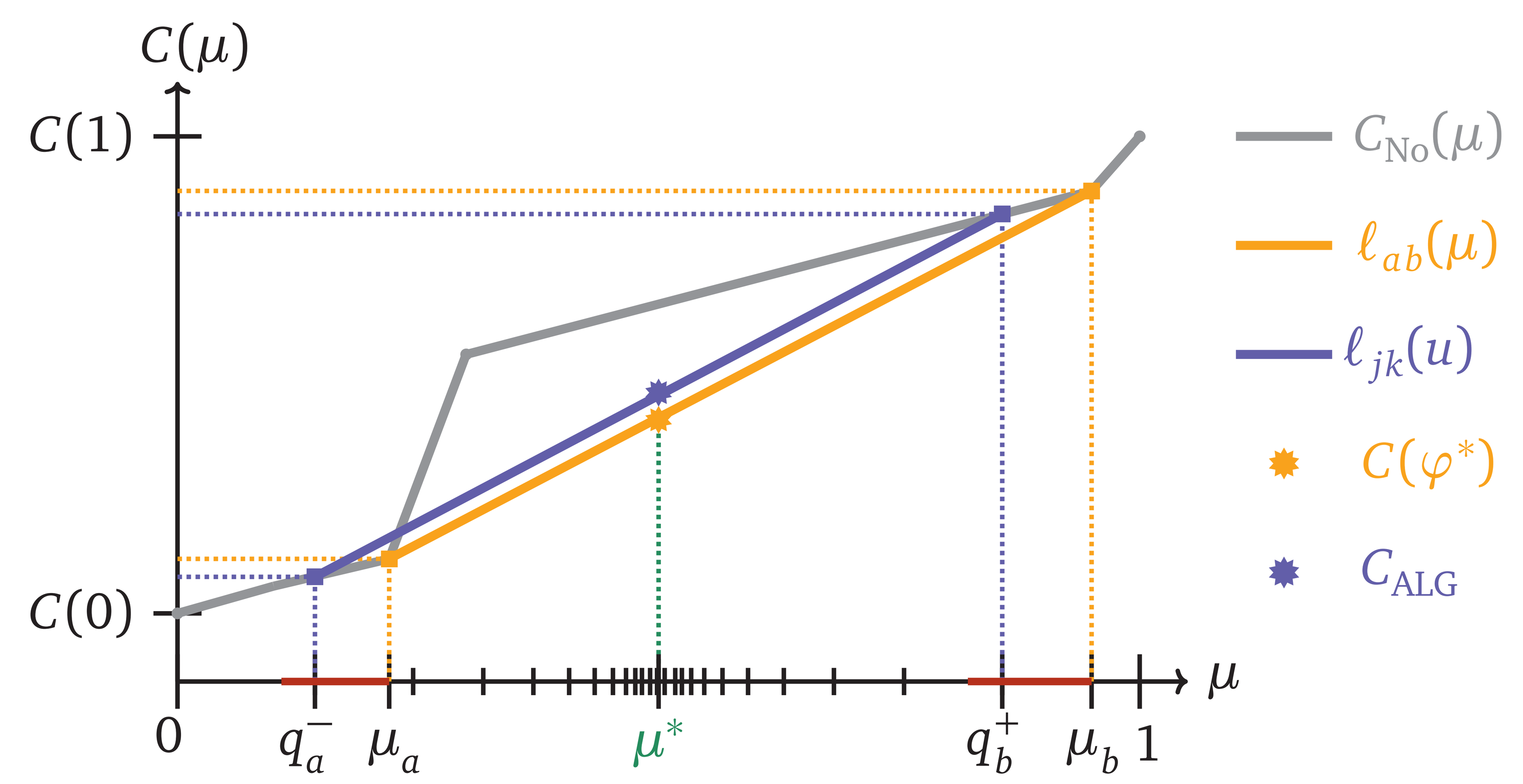
- $C(\mu)$ is piecewise linear due to system $\mathscr{L}$ of linear equations and inequalities that must be satisfied in WE:

$$\pi_v + a_e\left(\sum_{\theta\in\Theta}\mu_\theta d_\theta^2\right)x_e + b_e\sum_{\theta\in\Theta}\mu_\theta d_\theta = \pi_w \qquad \forall e \in A, \qquad (1)$$
$$\sum_{e\in\delta^+(v)}\sum_{\theta\in\Theta}\mu_\theta d_\theta^2 x_e - \sum_{e\in\delta^-(v)}\sum_{\theta\in\Theta}\mu_\theta d_\theta^2 x_e = \beta_v \qquad \forall v \in V, \qquad (2)$$
$$\pi_s = 0 \qquad (3)$$
$$\pi_v + a_e\left(\sum_{\theta\in\Theta}\mu_\theta d_\theta^2\right)x_e + b_e\sum_{\theta\in\Theta}\mu_\theta d_\theta \geq \pi_w \qquad \forall e \in E \setminus A, \qquad (4)$$
$$x_e \geq 0 \qquad \forall e \in E \qquad (5)$$

- $C(\mu)$ is non-decreasing if $|\Theta| = 2$.

- If $|\Theta| = 2$, there exists a fully polynomial-time approximation scheme (FPTAS) for computing φ^*:
  - The algorithm (ALG) computes polynomial many sample points $q_j^- < \mu^* < q_k^+$ for $C(\mu)$ with exponentially decreasing step size towards μ^*.
  - ALG determines the pair (q_j^-, q_k^+) for which the line $\ell_{jk}(\mu)$ through $C(q_j^-)$ and $C(q_k^+)$ is minimal at μ^* (or chooses No Signal).
  - Approximation ratio of $(1+\varepsilon)$ for any $\varepsilon > 0$ since there is always a sample point q in ε-distance (red area) to optimal μ_σ with lower cost.

  Illustration of ALG and proof: (again, $\Sigma = \{a, b\}$ and $\mu := \mu_{\theta_2}$)



$$C_{\text{ALG}} \leq \ell_{jk}(\mu^*) \leq \frac{\mu^* - q_a^-}{q_b^+ - q_a^-} C(\mu_b) + \frac{q_b^+ - \mu^*}{q_b^+ - q_a^-} C(\mu_a)$$

- Strict characterization: FI is always an optimal signaling scheme if and only if G is series-parallel.

- Given k distinct supports $(A_\sigma)_{\sigma\in[k]}$, the best signaling scheme inducing WE with supports $(A_\sigma)_{\sigma\in[k]}$ can be computed by an $\mathscr{L}$-based LP in time polynomial in $|\Theta|$, $|E|$, and k.

- Computational studies indicate that k is comparatively small for real-world traffic networks of various dimensions.

Network	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert Z\rvert$	d_{θ_2}
Sioux Falls (SF)	24	76	24	360,600
Eastern Massachusetts (EM)	74	258	74	65,576
Berlin-Friedrichshain (BF)	224	523	23	11,205
Berlin-Pr.-Berg-Center (BP)	352	749	38	16,660
Berlin-Tiergarten (BT)	361	766	26	10,755
Berlin-Mitte-Center (BM)	398	871	36	11,482

Network	k AV	k SD	k MAX	$C(\mu)$ concave [%]	$C(\mu)$ linear [%]
SF	4.67	2.08	9	80	10
EM	5.15	3.14	12	70	8
BF	5.28	2.76	12	68	10
BP	4.90	1.85	11	88	3
BT	5.10	2.54	11	78	8
BM	5.15	2.38	11	75	3

arXiv version: