Meeting times of Markov chains via singular value decomposition

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Introduction

- **Sequence of (possibly random) undirected graphs:** $G_n = (V_n := [n], E_n)$
- **Pair of independent discrete time random walks** on G_n: $(X_t^{(n)}, Y_t^{(n)})_{t\in\mathbb{Z}_+}$
- **Q: How long does it take them to meet?**

$$\tau^{(n)}_{\text{meet}}(i,j) = \inf\{t\in\mathbb{Z}_{\geq 0}: X_t^{(n)}=Y_t^{(n)}, X_0^{(n)}=i, Y_0^{(n)}=j\}$$
$$t^{\pi}_{\text{meet}} = \mathbf{E}_{(i,j)\sim\pi\otimes\pi}[\tau^{(n)}_{\text{meet}}(i,j)].$$

- **Applications:** Voter model, consensus time, algorithms on networks.

Materials and Methods

- **Generator:** $L := I_{n^2} - (P\otimes P)$.
- **Diagonal:** $\mathcal{D} = \{1, n+2, 2n+3, \ldots, n^2\}$.
- **Meeting matrix:** $E_{ij} := \mathbb{1}\{i=j\}\mathbb{1}\{i\in\mathcal{D}\}, \quad i,j\in[n^2]$,
- **Diagonally killed generator:** $L_{\text{kill}} := I_{n^2} - (P\otimes P)E$.
- **Singular value decomposition** of L and L_{kill}:
  - **Singular values of** L; L_{kill}: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n^2} = 0$; $\widetilde{\sigma}_1 \geq \widetilde{\sigma}_2 \geq \ldots \geq \widetilde{\sigma}_{n^2} > 0$.
  - **Singular vectors (L & R):** u_i and v_i; $\widetilde{u}_i$ and $\widetilde{v}_i$.
- **Non-asymptotic perturbation theory for SVD:**
  - $L_{\text{kill}} = L + \underbrace{(L_{\text{kill}} - L)}_{\text{perturbation}}$.
  - $\widetilde{\sigma}_i = \sigma_i + ?$, $\widetilde{u}_i = \widetilde{u}_i + ?$, $\widetilde{v}_i = \widetilde{v}_i + ?$

Conclusion

- Estimate the meeting time of two indedendent random walks on graphs via SVD of the diagonally killed generator of the pair
- View L_{kill} as a perturbation of L.
- Use rank one approximation of meeting times.
- Derive sharp non-asymptotic perturbation bounds of the rank one approximation for sufficiently dense Erdős–Rényi random graph.
- Outlook: Less dense random graphs.

References

[1] Thomas van Belle and Anton Klimovsky. Meeting times of markov chains via singular value decomposition. *arXiv preprint arXiv:2406.04958*, 2024.

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Expected meeting time via SVD

Proposition 1 (Expected meeting times via SVD of the diagonally killed generator)**.** *It holds that*

$$\text{vec}(\mathbf{E}[\tau^{(n)}_{\text{meet}}(i,j)]_{i,j\in[n]}) = E(I_{n^2} - (P\otimes P)E)^{-1}\underline{\mathbf{1}}_{n^2},$$

and in particular

$$t^{\pi}_{\text{meet}} = (\pi\otimes\pi)^t E(I_{n^2} - (P\otimes P)E)^{-1}\underline{\mathbf{1}}_{n^2}$$
$$= -1 + \sum_{i=1}^{n^2} \frac{1}{\widetilde{\sigma}_i}(\pi\otimes\pi)^t \widetilde{v}_i \widetilde{u}_i^t \underline{\mathbf{1}}_{n^2},$$

where $\underline{\mathbf{1}}_{n^2}$ is the n^2-dimensional vector with 1 at all coordinates, π is the stationary distribution.

Rank-k approximation of the expected meeting time:

$$\hat{t}^{\pi,(k)}_{\text{meet}} := -1 + \sum_{i=n^2-k+1}^{n^2} \frac{1}{\widetilde{\sigma}_i}(\pi\otimes\pi)^t \widetilde{v}_i \widetilde{u}_i^t \underline{\mathbf{1}}_{n^2}.$$

Proposition 2 (Rank-k approximation of the meeting time)**.** *It holds that*

$$\left|\hat{t}^{\pi,(k)}_{\text{meet}} - t^{\pi}_{\text{meet}}\right| \leq \frac{n\|\pi\|^2}{\widetilde{\sigma}_{n^2-k}}, \quad k\in[n^2].$$

Meeting on sufficiently dense Erdős–Rényi graphs

- **Erdős–Rényi random graph** $\mathcal{G}(n,p)$**:** $p\in(0,1)$, $n\in\mathbb{N}$ with symmetric adjacency matrix $A = (a_{i,j})_{i,j\in[n]}$, where $(a_{i,j})_{i<j}$ are independent Bernoulli(p) distributed entries; $a_{i,i} = 0$, $i\in[n]$; $a_{i,j} = a_{j,i}$, $i,j\in[n]$

Theorem 1 (Meeting time for sufficiently dense Erdős–Rényi graphs)**.** *Consider two independent simple random walks on Erdős–Rényi random graph $\mathcal{G}(n,p_n)$ with the edge probability*

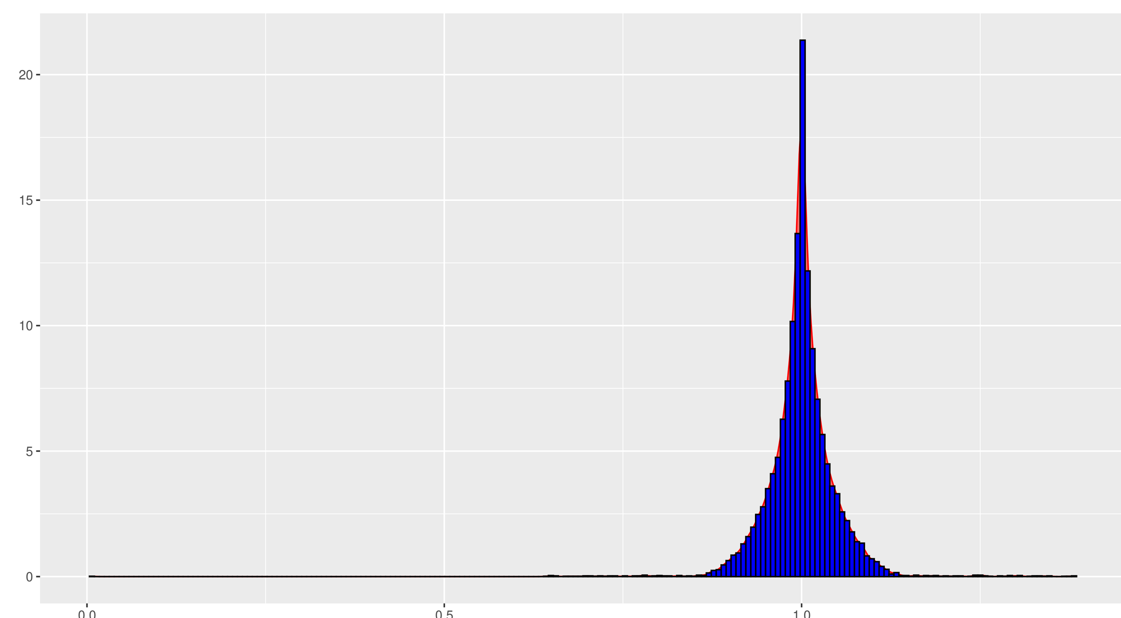
$$p_n := cn^{\beta-1}\wedge 1,$$

where $c>0$ arbitrary but fixed. Let $\beta > \frac{1}{2}$. Then, for all $\epsilon>0$, there exist $N\in\mathbb{N}$ and constants $\nu_1, \nu_2, \theta > 0$ such that for all $n\geq N$,

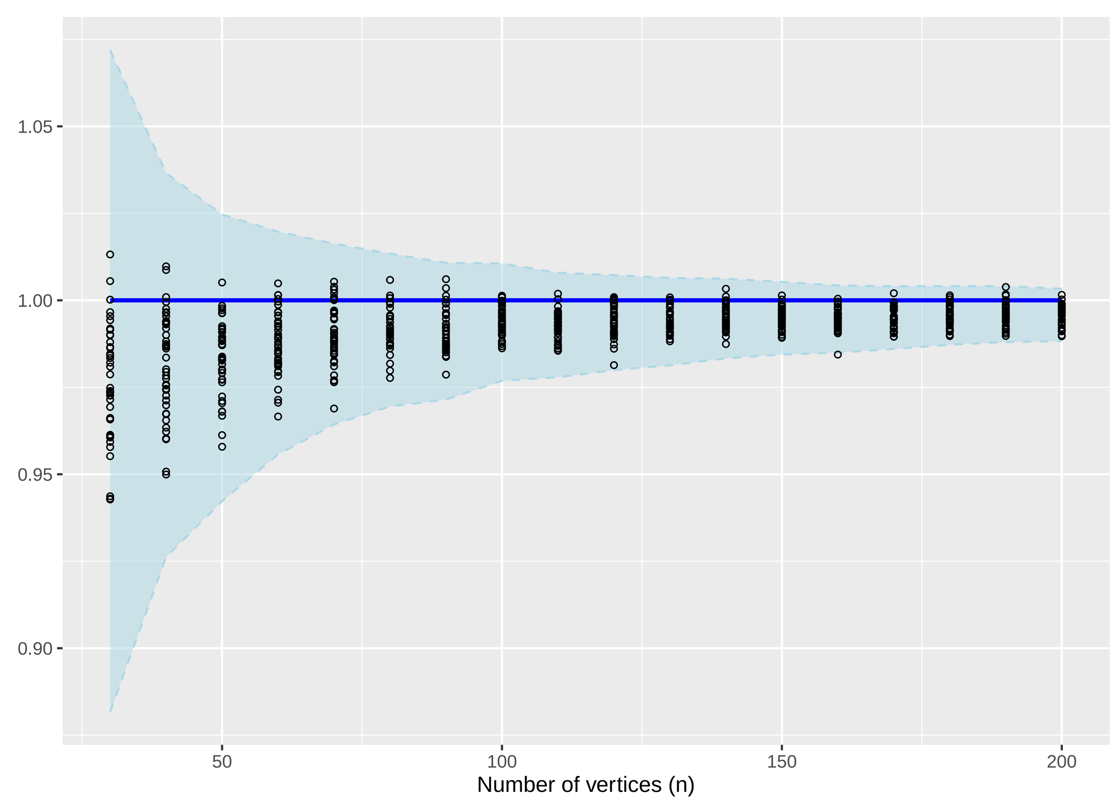
$$\mathbb{P}\left(\left|\frac{1}{n}t^{\pi}_{\text{meet}} - 1\right| > \epsilon\right) \leq 2n\exp\left(-\frac{\nu_1^2 d}{3}\right) + 2\binom{n}{2}\exp\left(-\frac{\nu_2^2 d^2}{3n}\right) + e^{-\theta(\log n)^2},$$

where $d := np_n$ is the mean degree. In particular,

$$t^{\pi}_{\text{meet}} \underset{n\to\infty}{\sim} n \text{ w.h.p.}$$



- The histogram of singular values of L_{kill}...
- for a reazation of the Erdős–Rényi random graph...
- with $n = 100$, $\beta = \frac{2}{3}$.



- rank one approximations (divided by n)...
- vs. the asymptotic value $= 1$