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FACULTY
OF MATHEMATICS, INFORMATICS
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UNDECIDED STATE DYNAMICS WITH STUBBORN AGENTS

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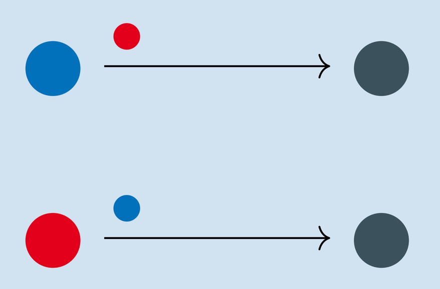
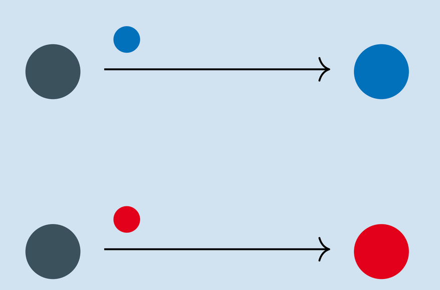
Model.

- Population protocol
  - n agents
  - Asynchronous pairwise interactions
- Based on Undecided State Dynamics (USD) [1]
  - Opinion 1,
  - Opinion 2,
  - Undecided state $\perp$.

- Op. i meeting Op. $j \neq i$ becomes undecided.



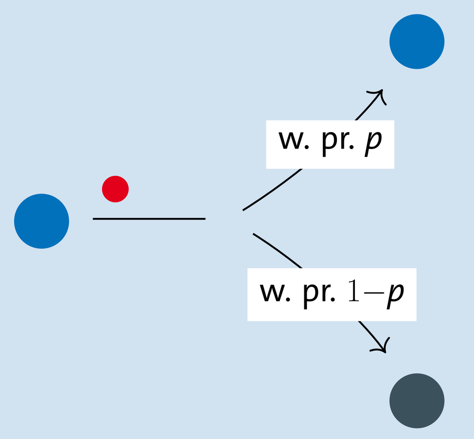
- Undecided agent meeting any op. adopts that opinion.



- No change when Op. i meets Op. i or $\perp$

- *Stubborn* Undecided State Dynamics
  - Opinion 1 is the *preferred* opinion
  - *Stubbornness parameter* $p \in [0, 1]$.

- Op. 1 meeting Op. 2 in the Stubborn USD

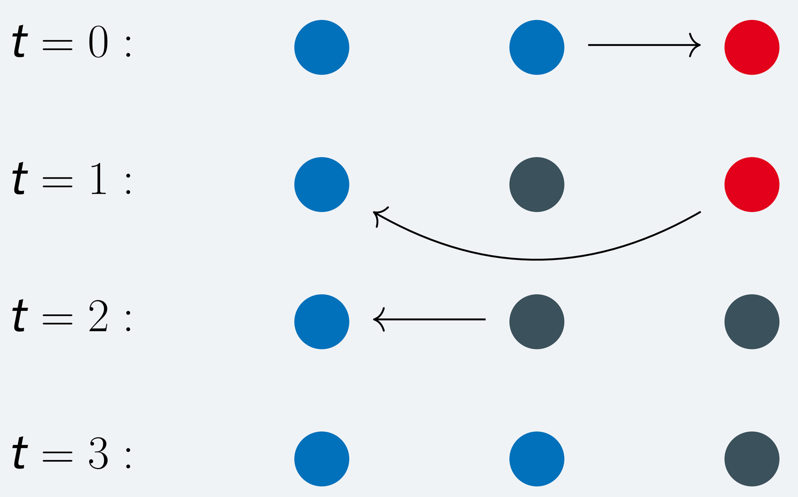


- Remaining interactions identical to USD

- Full transition function:

$$(q, q') \mapsto \begin{cases} \perp \text{ if } q = 2, q' = 1 \\ \perp \text{ if } q = 1, q' = 2 \text{ w. pr. } 1 - p \\ q' \text{ if } q = \perp \\ q \text{ otherwise.} \end{cases}$$

- Example interactions



Results.

- Agents eventually agree on one opinion
- $T_i(p, \mathbf{x})$: time until agents agree on Opinion i when starting from configuration $\mathbf{x}$ and with stubbornness parameter p
- **Goal: Find bounds for** $T_i(p, \mathbf{x})$.
- x_i: (initial) number of agents with Opinion i.
- u: (initial) number of undecided agents.
- **Phase transition** for p at $p_s := 1 - x_1/x_2$.

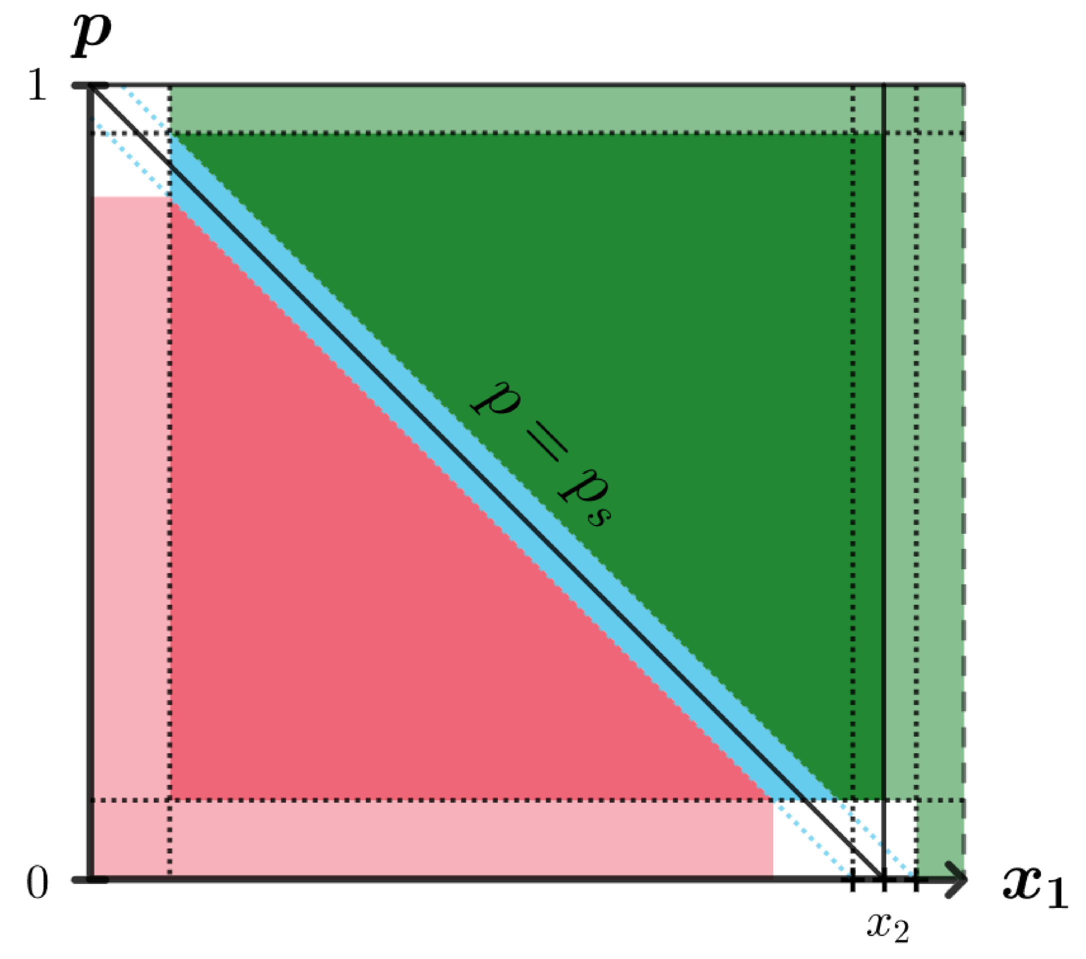
Theorem 1. Let $\epsilon, p \in (0, 1]$ be arbitrary constants and let $\mathbf{x} = (x_1, x_2, u)$ be a configuration with $x_1 \in [\epsilon \cdot n, x_2]$, $u \leq \frac{n}{2}$. Then w.h.p.

$$T_1(p, \mathbf{x}) = O(n \cdot \log n) \ \text{ if } p - p_s = \Omega\left(\sqrt{n^{-1} \cdot \log n}\right),$$
$$T_2(p, \mathbf{x}) = O(n \cdot \log n) \ \text{ if } p_s - p = \Omega\left(\sqrt{n^{-1} \cdot \log n}\right),$$
$$T_{1 \vee 2}(p, \mathbf{x}) = O(n \cdot \log^2 n) \text{ otherwise.}$$

Theorem 2. Let $p, \tilde{p} \in [0, 1]$ and let $\mathbf{x} = (x_1, x_2, u), \tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{u})$ be arbitrary configurations. Then, it holds for all $t \geq 0$ that

$$\Pr[T_1(p, \mathbf{x}) \leq t] \geq \Pr[T_1(\tilde{p}, \tilde{\mathbf{x}}) \leq t]$$
$$\text{if } \tilde{p} \leq p, \tilde{x}_1 \leq x_1 \text{ and } \tilde{x}_2 \geq x_2,$$
$$\Pr[T_2(p, \mathbf{x}) \leq t] \geq \Pr[T_2(\tilde{p}, \tilde{\mathbf{x}}) \leq t]$$
$$\text{if } \tilde{p} \geq p, \tilde{x}_1 \geq x_1 \text{ and } \tilde{x}_2 \leq x_2.$$

Winning Opinion based on p and Initial Configuration.



- Dashed blue lines around black diagonal represent $p = p_s \pm \Theta(\sqrt{n^{-1} \cdot \log n})$.
- At $p = 0$ coincidence with known bounds from [2].
- Inner opaque rectangular area corresponds to Theorem 1.
- Remaining colored areas correspond to Theorem 2.
- Green area: Opinion 1 wins w.h.p. in $O(n \log n)$ interactions.
- Red area: Opinion 2 wins w.h.p. in $O(n \log n)$ interactions.
- Blue area: either Opinion wins w.h.p. in $O(n \log^2 n)$ interactions.

Analysis.

- Define **weighted bias** $\Delta_w(t) = x_1(t) - (1 - p)x_2(t)$.
  - Note: $\Delta_w(0) = 0 \Leftrightarrow p = p_s$.

Equilibrium configurations.

$$\mathrm{E}[\Delta_w(t + 1) \mid \Delta_w(t) = 0] = 0 \qquad \text{(independent of } u\text{)}$$

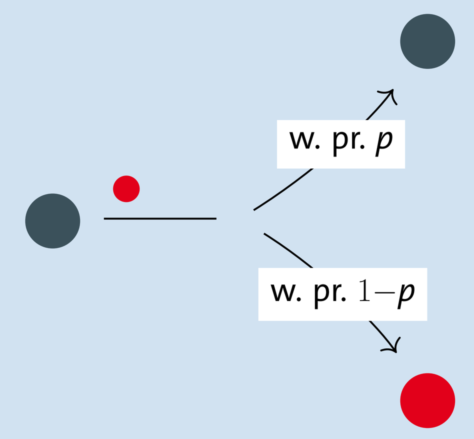
We show the following (with high probability):

- Creation of sufficient weighted bias in $O(n \log^2 n)$ interactions if $|\Delta_w(0)| = o(\sqrt{n \log n})$
- Repeated doubling of the weighted bias until $x_1 = 0$ or $x_2 = 0$ in $O(n \log n)$ interactions (drift)
- Convergence to $x_1 = n$ or $x_2 = n$ in $O(n \log n)$ interactions
- Coupling results (Theorem 2) for many initial configurations
- *Note:* Lower bound of $\Omega(n \log n)$ to even activate all agents once

Open Questions.

- $O(n \log n)$ for $|\Delta_w(0)| = o(\sqrt{n \log n})$?
- Analyze *stubborn undecided* variant of the USD

- All but one interaction as in the USD



- Drift depends on initial number of undecided agents

References.

[1] D. Angluin, J. Aspnes, and D. Eisenstat, "A simple population protocol for fast robust approximate majority," *Distributed Computing*, vol. 21, no. 2, pp. 87–102, 2008. DOI: 10.1007/s00446-008-0059-z.

[2] A. Condon, M. Hajiaghayi, D. G. Kirkpatrick, and J. Manuch, "Simplifying analyses of chemical reaction networks for approximate majority," in *DNA Computing and Molecular Programming - 23rd International Conference, DNA 23*, ser. Lecture Notes in Computer Science, vol. 10467, Springer, 2017, pp. 188–209. DOI: 10.1007/978-3-319-66799-7_13.

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