Image Segmentation with Bregman Split

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Introduction

We describe an implementation of the two-phase image segmentation algorithm. This algorithm partitions the domain of a given 2d image into foreground and background regions, and each pixel of the image is assigned membership to one of these two regions. The underlying assumption for the segmentation model is that the pixel values of the input image can be summarized by two distinct average values, and that the region boundaries are smooth.

Segmentation Model

The model aims to segment the domain $\Omega \subset \mathbb{R}^2$ of a grayscale image $f : \Omega \to \mathbb{R}$ into a foreground region Ω_1, with perimeter $\mathrm{Per}(\Omega_1)$ and a background region $\Omega_2 = \Omega \setminus \Omega_1$. The image values in Ω_1, Ω_2 can be approximated by averages c_1, c_2 respectively.This model can be formulated as the minimization of the energy:

$$\mathrm{Per}(\Omega_1)+\lambda\int_{\Omega_1}(f(x)-c_1)^2dx+\lambda\int_{\Omega_2}(f(x)-c_2)^2dx$$

Theorem [3]: For any given fixed $c_1, c_2 \in \mathbb{R}$, a global minimizer for the above energy is obtained by solving the convex minimization: $\min_{0\le u\le 1}\int_\Omega |\nabla u|dx + \lambda\int_\Omega \left\{(f(x)-c_1)^2-(f(x)-c_2)^2\right\}u(x)dx \quad (1)$ and then setting $\Omega_1 = \{x : u(x) \ge \mu\}$ for a.e. $\mu \in [0,1]$.

Bregman Iterations

Suppose J and H are (possibly non-differentiable) convex functionals defined on a Hilbert space X. The Bregman iterations can be used to solve a convex minimization problem of the form $\min_{u\in B} J(u) + \lambda H(u),\ \lambda > 0$.

Suppose $J : X \to \mathbb{R}$ is a convex function and $u \in X$. An element $p \in X^*$ is called a subgradient of J at v if for all $u \in X$: $J(u) - J(v) - \langle p, u-v\rangle \ge 0$. The set of all subgradients of J at v is called the subdifferential of J at v, and it is denoted by $\partial J(v)$.

Suppose $J : X \to \mathbb{R}$ is a convex function, $u, v \in X$ and $p \in \partial J(v)$. Then the Bregman distance between points u and v is defined by: $D_J^p(u,v) := J(u) - J(v) - \langle p, u-v\rangle$

Bregman iterations for differentiable H are summarized below.

initialization: $p^0 \in \partial J(u^0)$

For $k = 0, 1, \dots$

$u^{k+1} = \mathrm{argmin}_u D_J^{p^k}(u, u^k) + \lambda H(u)$

$p^{k+1} = p^k - \lambda\nabla H(u^{k+1})$

We are interested in Bregman iterations in the case of linear constraints $Au = z$, :$\min_u J(u) + \frac{\lambda}{2}\|Au - z\|^2$. The iterations can be shown to be equivalent to:

initialization: $u^0, b^0 = 0$, z is given data

For $k = 0, 1, \dots$ $u^{k+1} = \mathrm{argmin}_u J(u) + \frac{\lambda}{2}\|Au - z - b^k\|^2$

$b^{k+1} = b^k + z - Au^k$

Discretized Model

We discretize the segmentation energy (1) weighted by function g as follows:

$$\sum_{i,j} g_{i,j}|\nabla u_{i,j}| + \lambda\sum_{i,j}((f_{i,j}-c_1)^2-(f_{i,j}-c_2)^2)u_{i,j}.$$

We introduce the auxiliary variable $d_{i,j} = \nabla u_{i,j}$, and consider the equivalent augmented Lagrangian.

$\mathrm{argmin}_{0\le u\le 1, d}\sum_{i,j} g_{i,j}|d_{i,j}| + \lambda\sum_{i,j}((f_{i,j} - c_1)^2 - (f_{i,j}-c_2)^2)u_{i,j} + \frac{\gamma}{2}\sum_{i,j}(d_{i,j} - \nabla u_{i,j} - b_{i,j})^2,\ \gamma > 0.$

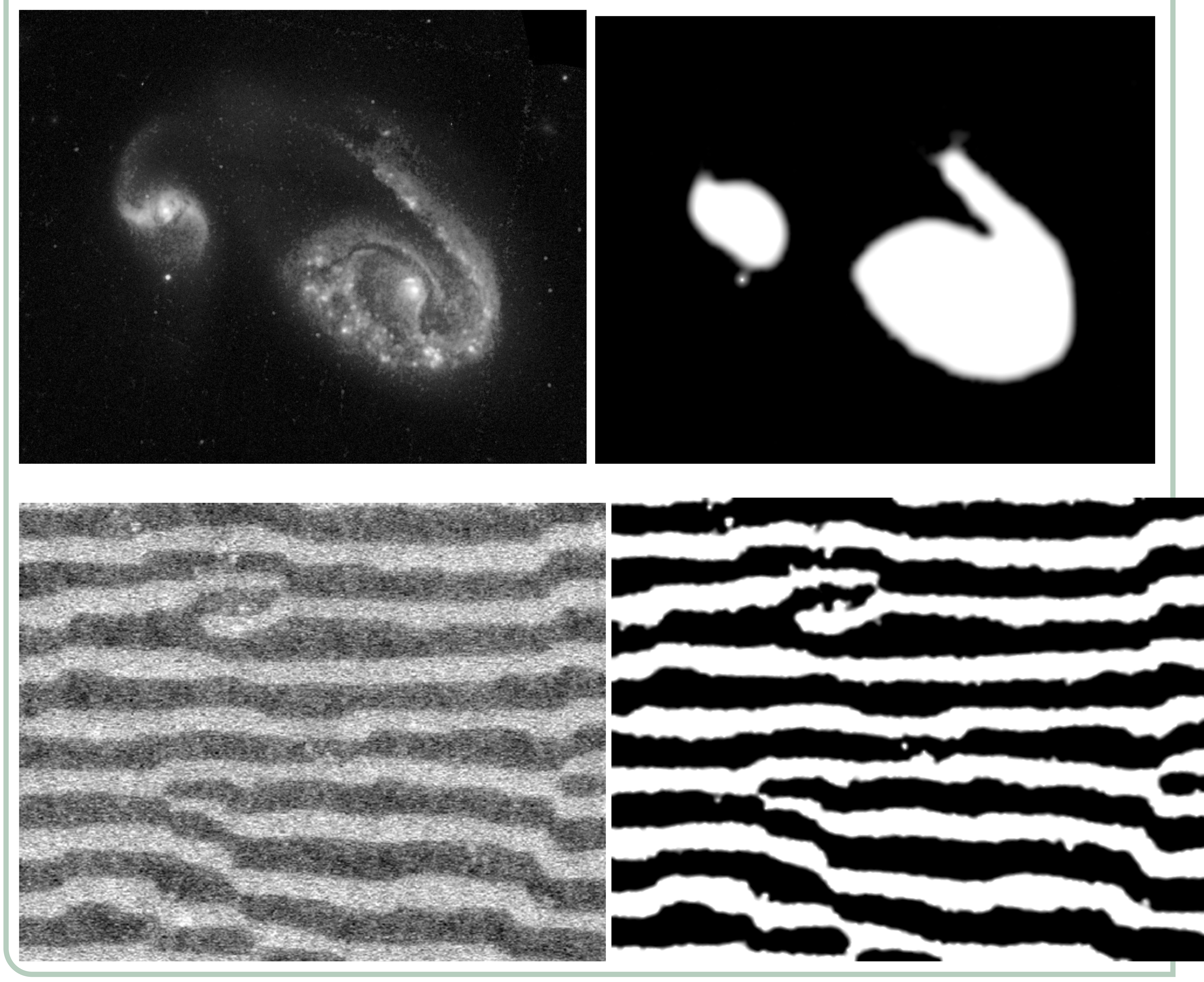
To minimize the discretized augmented Lagrangian, we minimize first over d, then over u while keeping the other variable fixed. We iterate in this manner until convergence.

Minimize discrete model

The d subproblem: $\mathrm{argmin}_d \sum_{i,j} g_{i,j}|d_{i,j}| + \frac{\gamma}{2}\sum_{i,j}(d_{i,j} - \nabla u_{i,j} - b_{i,j})^2$ with solution: $d_{i,j} = \frac{\nabla u_{i,j}+b_{i,j}}{|\nabla u_{i,j}+b_{i,j}|}\max\{|\nabla u_{i,j} + b_{i,j}| - \frac{g_{i,j}}{\gamma}, 0\}$

The u subproblem:
$\mathrm{argmin}_{0\le u\le 1}\lambda\sum_{i,j}((f_{i,j} - c_1)^2 - (f_{i,j} - c_2)^2)u_{i,j} + \frac{\gamma}{2}\sum_{i,j}(\nabla u_{i,j} - d_{i,j} + b_{i,j})^2$.
The optimal u satisfies $\Delta u_{i,j} = \frac{\lambda}{\gamma}r_{i,j} + \mathrm{div}(d_{i,j} - b_{i,j})$ where $r_{i,j} = (f_{i,j} - c_1)^2 - (f_{i,j} - c_2)^2$.
The solution to the **u-subproblem** can be obtained with Gauss-Seidel iterations
Region update: $\Omega_1^k = \{(i,j) : u_{i,j}^k \ge 0.5\}, \Omega_2^k = \{(i,j) : u_{i,j}^k \le 0.5\}$; and $c_1^k = \frac{1}{|\Omega_1^k|}\sum_{(i,j)\in\Omega_1^k} f_{i,j}, c_2^k = \frac{1}{|\Omega_2^k|}\sum_{(i,j)\in\Omega_2^k} f_{i,j}$.

Experiments and Results



Further Direction

- Apply this iteration technique to other optimization problems like compressed sensing.
- Image segmentation of 3d images and multiphase segmentation.

References

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