Hitting time of perfect matchings in Cartesian product graphs

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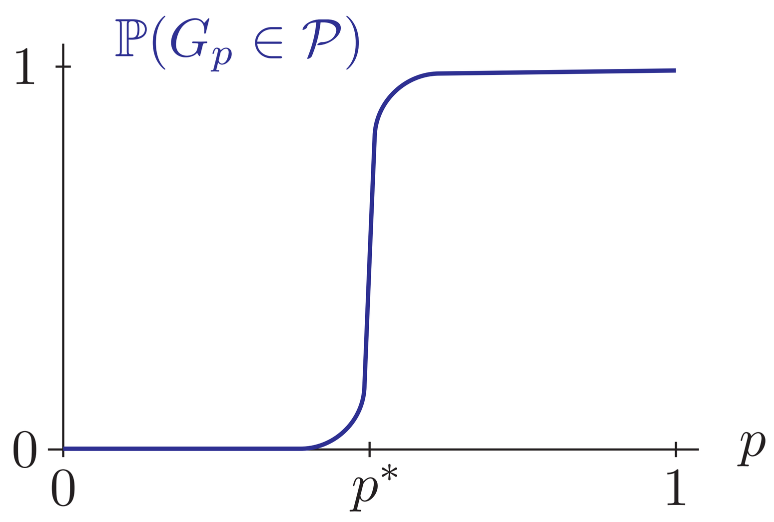
Thresholds

Threshold phenomena arise naturally in many contexts and have received a lot of attention in physics as well as in mathematics. Looking at a fixed graph property $\mathcal{P}$ of graphs, we ask:

What is the probability that a random graph G_p has property $\mathcal{P}$?

Increasing the probability p makes the graph denser. If the property $\mathcal{P}$ is non-empty and increasing, there is a critical probability p^* around which there is a drastic change: the probability of having property $\mathcal{P}$ jumps from zero to one [2].



This sharp transition is called a **threshold phenomenon**.

Hitting times

Given a graph $G = (V, E)$

1) the **random graph** G_p is on vertex set V where each edge of G is retained with probability p independently.
2) the **random graph process** on G starts with the empty graph $G(0)$ and at each step $1 \leq i \leq |E|$, $G(i)$ is obtained from $G(i-1)$ by adding uniformly at random a new edge from E.

The **hitting time** of a monotone increasing (non-empty) graph property $\mathcal{P}$ is a random variable equal to the minimum index τ for which $G(\tau) \in \mathcal{P}$, but $G(\tau - 1) \notin \mathcal{P}$.
If two properties $\mathcal{P}_1, \mathcal{P}_2$ have the same hitting time, they come up at exactly the same time in the random graph process. Having the same hitting time is an even stronger property than having the same threshold.

Graph properties

Three very well-studied properties of graphs are

- $\mathcal{P}_C$ connectedness,
- $\mathcal{P}_D$ minimum degree one,
- $\mathcal{P}_{PM}$ existence of a perfect matching.

Note that deterministically
$$\tau(P_C) \geq \tau(P_D) \text{ and } \tau(P_{PM}) \geq \tau(P_D).$$

References

[1] B. Bollobás. Complete matchings in random subgraphs of the cube. *Random Structures Algorithms*, 1(1):95–104, 1990.

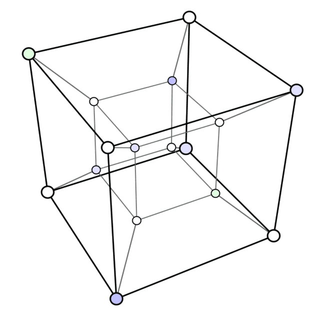
[2] B. Bollobás and A. Thomason. Threshold functions. *Combinatorica*, 7(1):35–38, 1987.

[3] S. Diskin, J. Erde, M. Kang, and M. Krivelevich. Isoperimetric inequalities and supercritical percolation on high-dimensional graphs. *Combinatorica*, 2024.

[4] S. Diskin and A. Geisler. Perfect matching in product graphs and in their random subgraphs. 2024.

Main results

In 1990, Bollobás [1] showed:



> ***Whp***, $\tau(P_C) = \tau(P_D) = \tau(P_{PM})$ *in the random graph process on the t-dimensional hypercube Q^t.*

We generalize this to regular Cartesian product graphs of bounded size base graphs.
Let $C > 1$. For every $i \in [t]$, let H_i be a d_i-regular connected graph with $1 < |V(H_i)| \leq C$. Let $G = \square_{i=1}^t H_i$ denote the Cartesian product graph of $H_1, \ldots, H_t$ (see below for definition).

> **Theorem 1.** *[4] **Whp**, $\tau(P_C) = \tau(P_D) = \tau(P_{PM})$ in the random graph process on G.*

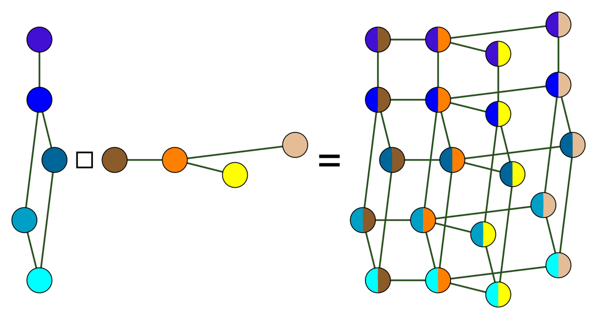
Product graphs

Given t graphs, $H_i, \ldots, H_t$, their Cartesian product $G = \square_{i=1}^t H_i$ is the graph with the vertex set

$$V := \{v = (v_1, \ldots, v_t) \colon v_i \in V(H_i) \text{ for all } i \in [t]\},$$

and the edge set

$$\left\{ uv \colon \begin{array}{l} \text{there is some } i \in [t] \text{ such that } u_j = v_j \\ \text{for all } i \neq j \text{ and } \{u_i, v_i\} \in E(H_i) \end{array} \right\}.$$



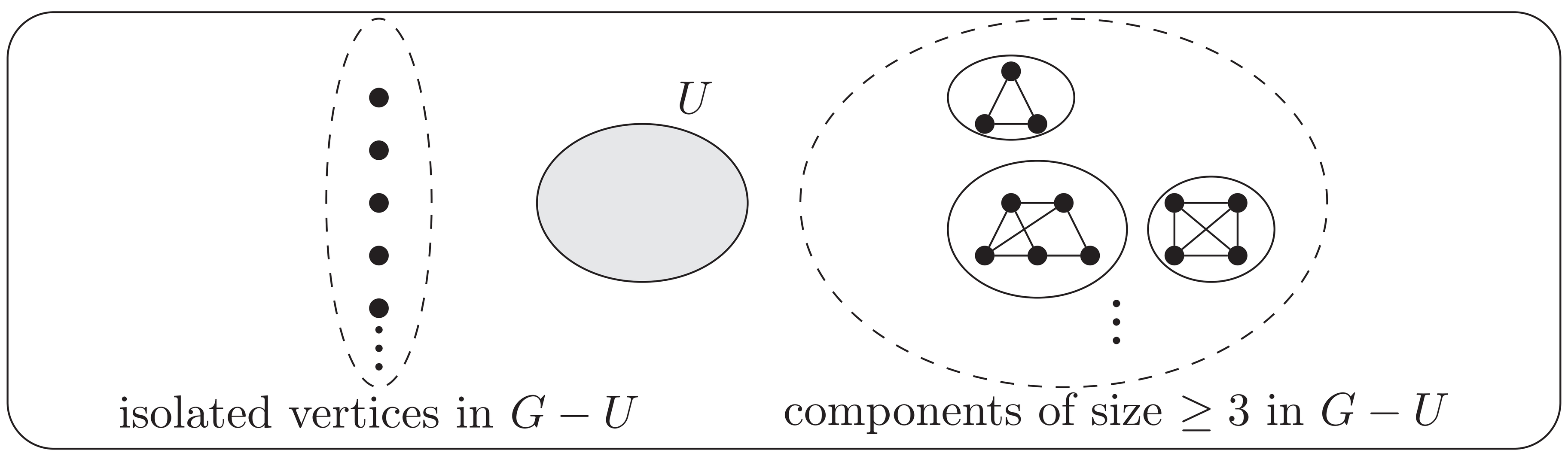
We call $H_1, H_2, \ldots, H_t$ the *base graphs of* G.
The hypercube is then the product of edges, namely $Q^t = \square_{i=1}^t K_2$.

Proof ideas

Consider p just before the threshold of minimum degree one. The following theorem characterizes the structure of G_p and implies Theorem 1.

> **Theorem 2.** *[4] Let $\epsilon \geq 0$ be a sufficiently small constant, and let p be such that $(1-p)^d \leq n^{-(1-\epsilon)}$. Then, **whp**, the following properties hold in G_p.*
>
> *(1) Every two isolated vertices in G_p are at distance at least two in G.*
>
> *(2) There exists a unique giant component, spanning all but $o(n)$ of the vertices. All the other components of G_p, if there are any, are isolated vertices.*
>
> *(3) The giant component of G_p has a perfect matching.*

Properties 2 and 2 follow from a classical **sprinkling argument**. Then adding any edge in the random graph process, it either lies in the giant component or connects an isolated vertex to the giant component. Thus, exactly at the point when the last isolated vertex disappears (at $\tau(P_D)$) the graph becomes connected (at $\tau(P_C)$). Furthermore by Property 2 the only obstructions to a perfect matching are the isolated vertices and thus at the same time as these disappear (at $\tau(P_D)$) there is a matching covering the graph (at $\tau(P_{PM})$).
Let us take a more detailed look at Property 2. Suppose there is no perfect matching in G_p, then there is a set of vertices U such that $G - U$ contains more than $|U|$ odd components **(Tutte's theorem)**.



For each component in $G - U$ all the edges leaving it, go to U. By the **expansion properties** ([3]) of regular product graphs, there are many edges leaving these components. On the other hand, since G is d-regular, there are at most $d|U|$ edges touching U. Thus many edges are not present in G_p, which is a low probability event.
The main challenge is to bound the number of possible choices for U, such that the union bound can be applied. In order to achieve this, we split into different cases according to the size of U and the structure of $G - U$. Using the **product structure** of the underlying graph allows to bound the number of choices for U, where $|U|$ is large and has low expansion.