Bootstrap percolation on the high-dimensional Hamming graph

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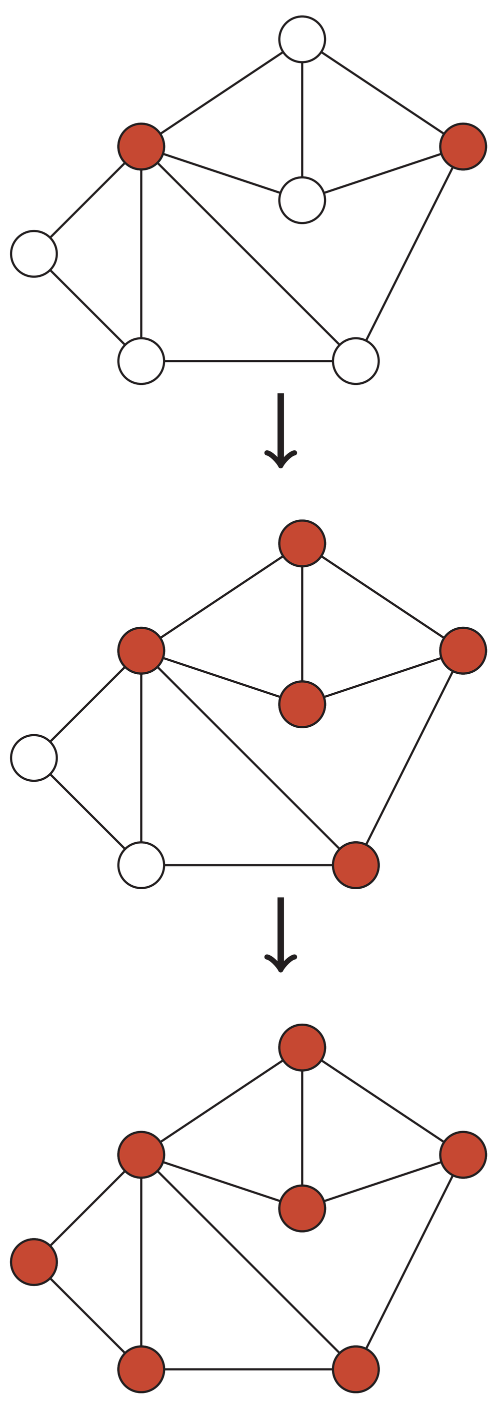
Bootstrap Percolation

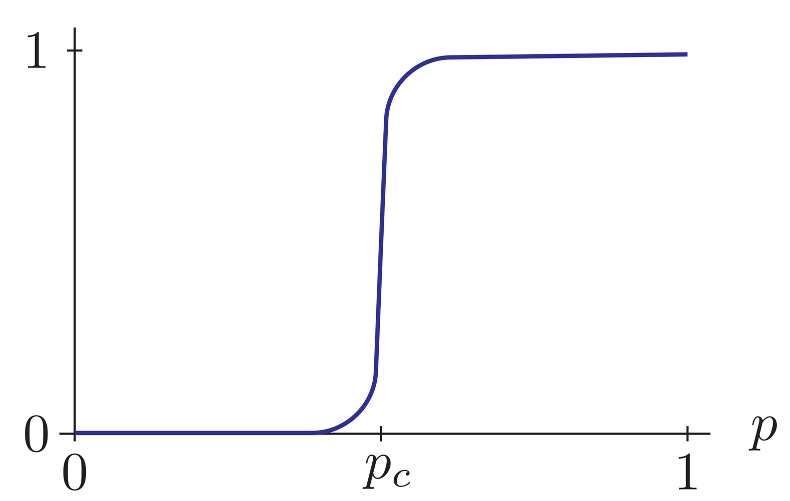
Bootstrap percolation was introduced in 1979 in the context of magnetic systems. It has many applications ranging from studying the spread of information to modelling collective behaviour. Given a graph G, a set $A_0 \subseteq V(G)$ of **initially infected** vertices and the so-called **infection parameter** $r \in \mathbb{N}$, a **healthy** vertex gets infected if it has at least r infected neighbours.



G **percolates** if eventually all vertices get infected.

Critical probability

In **random** bootstrap percolation, the set of **initially infected** vertices is given as a random subset $\boldsymbol{A}_p \subseteq V(G)$, where each vertex is retained independently with probability $p \in (0, 1)$. For many graphs, a **threshold phenomenon** is observable, where for increasing values of p, the probability of G percolating undergoes a drastic change from being almost 0 to being almost 1.



The **critical probability** $p_c(G)$ is the point at which the probability that G percolates passes through 1/2.

$$p_c(G, r) := \inf \{p \mid \mathbb{P}[G \text{ percolates }] \geq 1/2\}.$$

References

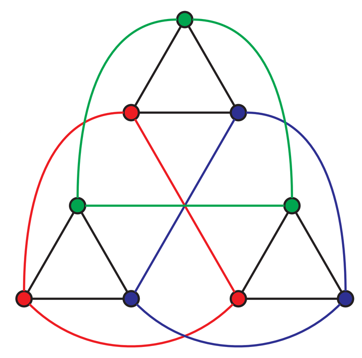
[1] J. Balogh and B. Bollobás. Bootstrap percolation on the hypercube. *Probab. Theory Related Fields*, 134(4):624–648, 2006.

[2] A. E. Holroyd. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields*, 125(2):195–224, 2003.

[3] M. Kang, M. Missethan, and D. Schmid. Bootstrap percolation on the high-dimensional hamming graph, 2024. arXiv:2406.13341 [math.CO].

The Hamming graph

Given $n, k \in \mathbb{N}$, the n-dimensional Hamming graph $G = \square_{i=1}^n K_K$ with base graphs K_k is the graph with vertex set $[k]^n$ where two vertices are adjacent if they differ in exactly one coordinate. In the special case $k = 2$, the Hamming graph $\square_{i=1}^n K_2$ is the n-dimensional hypercube Q^n.



Main results

In 2006, Balogh and Bollobás[1] established a threshold for random bootstrap percolation on the n-dimensional hypercube with infection parameter $r = 2$:

Theorem ***Whp***, $p_c(Q^n, 2) = \Theta(1)\, n^{-2} 2^{-2\sqrt{n}}$.

We generalize this result to the Hamming graph.
Let $n, k \in \mathbb{N}$ satisfy $2 \leq k \leq 2^{\sqrt{n}}$. Consider random bootstrap percolation on the n-dimensional Hamming graph $G = \square_{i=1}^n K_k$ with infection parameter $r = 2$.

Theorem 1 ([3])**.** ***Whp***, $p_c(\square_{i=1}^n K_k, 2) = \Theta(1)\, n^{-2} k^{-2\sqrt{n}+1}$.

The lower threshold

A **projection** of G is a subgraph of G that is isomorphic to a lower-dimensional Hamming graph (e.g., in the case $G = Q^n$ a projection is a subcube of G). A projection is **internally spanned**, if the subgraph induced by the projection percolates. Using so-called 'hierarchy' methods, which were introduced by Holroyed [2], we can show roughly that in order for G to percolate, there has to exist an internally spanned projection of every dimension in G (up to error terms of order $\Theta(k)$).

Theorem 2. *For all $t \in \mathbb{N}$ the following holds.*

$$\mathbb{P}[G \textit{ percolates}] \leq \mathbb{P}[\textit{there exists an internally spanned projection of dimension } t].$$

Similarly, we can derive upper bounds on the probability that a projection is internally spanned. Let us denote by X_t the number of internally spanned projections of dimension t in G. We show that for a certain critical dimension t_* the expected number of internally spanned projections turns to 0.

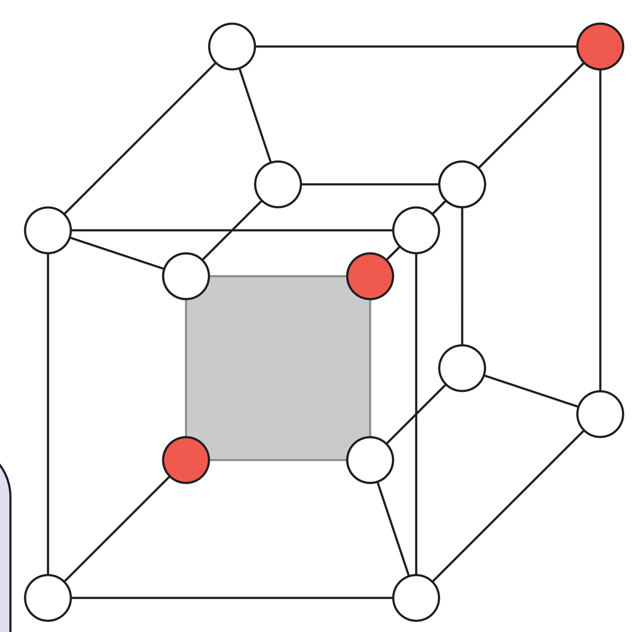
Theorem 3. *Set $t_* = 2\sqrt{n} - 2$. Then,*

$$\mathbb{E}[X_{t_*}] = o(1).$$

By first moment method, the probability there is an internally spanned projection of dimension t_* is bounded from above by its expected number. So, Theorems 2 and 3 imply the lower threshold.

The upper threshold

A sequence $v = (v_0, \dots, v_\ell)$ of vertices is called **sequentially spanning**, if it satisfies the property that for any j, the subsequence $(v_0, \dots, v_j)$ spans a projection of dimension $2j$ in G. Crucial properties of such sequences are that they are minimum percolating and that a majority of all minimum percolating sets is of this type. Using the second moment method, show the existence of a sequence $(v_0, \dots, v_{\frac{n}{2}})$, where each vertex was initially infected.



Theorem 4. *Denote by Y the number of i.s.s. sequences. If*

(a) $\mathbb{E}[Y] \to \infty$;

(b) $\mathbb{V}[Y] = o\left(\mathbb{E}[Y]^2\right)$.

Then $\mathbb{P}[Y \geq 1] \to 1$.

To show (a), recursively count ways to extend sequentially spanning sequences of given length.
To show (b), for each j, count pairs of such sequences that share j vertices.
Then, by definition, the sequence $(v_0, \dots, v_{\frac{n}{2}})$ forms a percolating set in G.