A Quantitative Version of More Capable Channel Comparison

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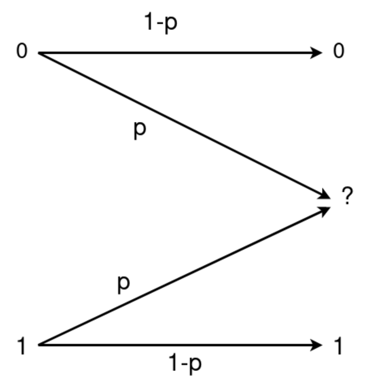
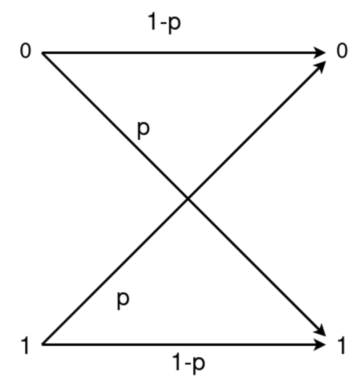
Introduction

Broadcast channels are a fundamental concept in information theory, used to model errors and distortions of data transmission. Two well-studied examples are called Binary Symmetric Channel (BSC) and the Binary Erasure Channel (BEC).



BEC



BSC

There are many notions of comparing two channels to each other. We introduce a quantitative generalization of the "more capable" channel comparison, termed "**more capable with advantage**". Subsequently, we present two situations in which this concept is applied.

The poster uses two key concepts in information theory: **entropy** and **mutual information**. Entropy, denoted by $H(X) := \sum_{x\in\mathcal{X}} p(x)\log_2 1/p(x)$, measures the randomness of a random variable X. Mutual information, denoted by $I(X;Y) := H(X)+H(Y)-H(X,Y)$, quantifies the dependence between two random variables X and Y.

Application 1: List Decoding

A **binary error-correcting code** of block length n is a subset of $\mathbb{F}_2^n$. It can also be equivalently defined in terms of an encoding and decoding process. The most critical parameter of a code is its rate, which governs the amount of redundancy it uses. This importance of the rate is evidenced by the **Shannon's coding theorem**, which provides precise rate limit (called **channel capacity**) for reliable communication through a given channel. When this rate is exceeded, unique decoding of errors is not possible. In that case, the decoding process can still be successful, if it outputs not just a single message but a list of messages of a given size. This process is called **list-decoding**.

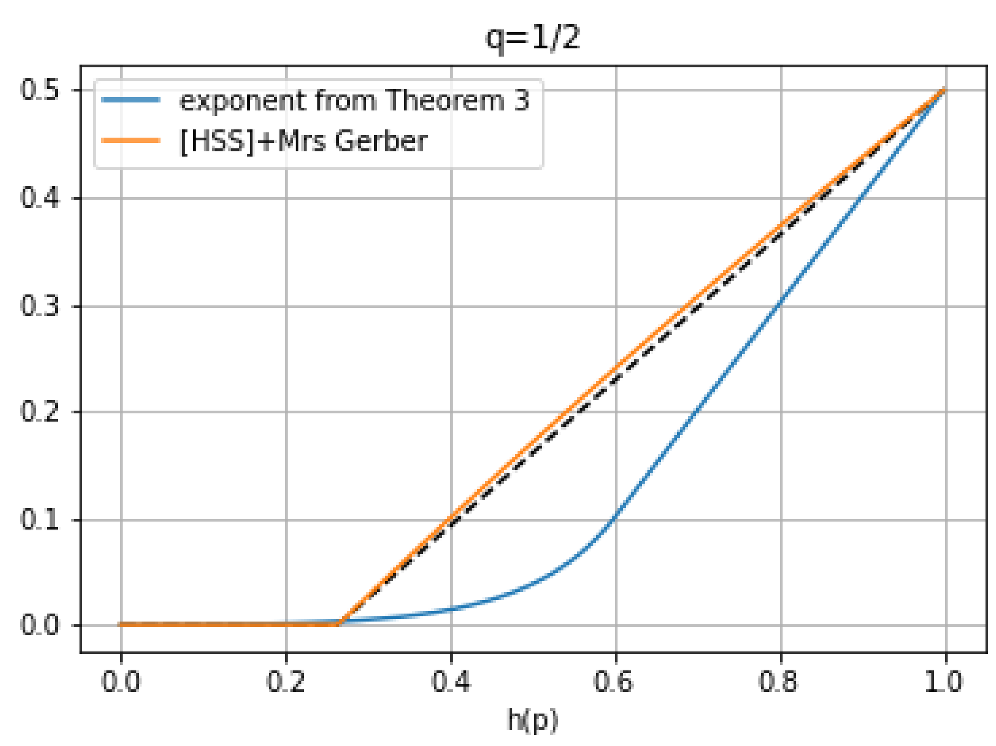
We are motivated by the following natural question:

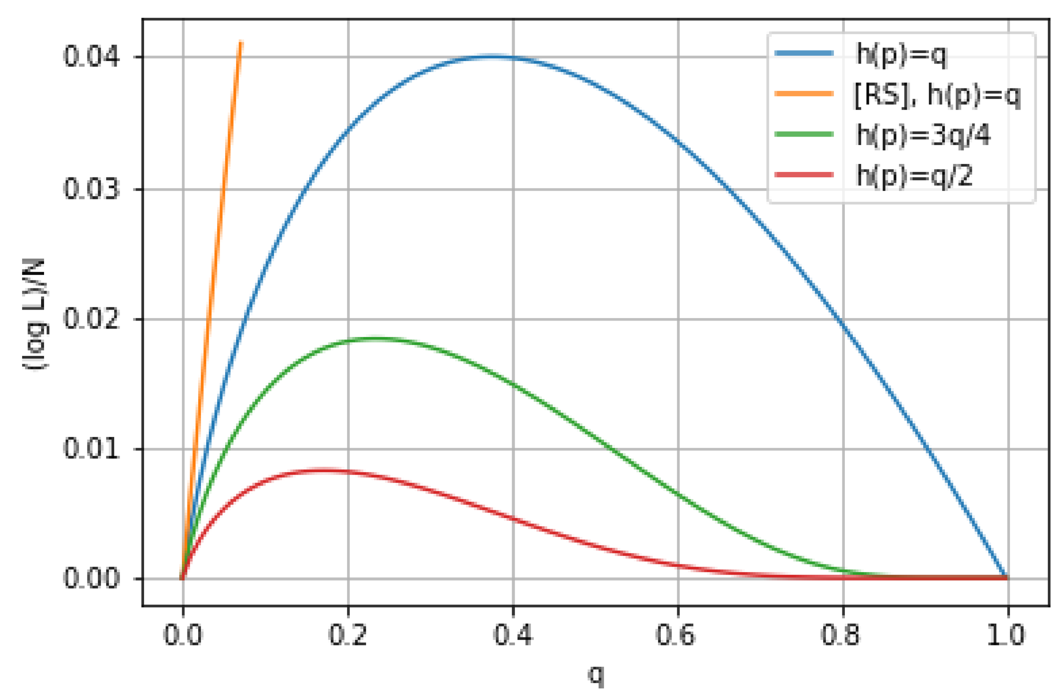
Let $0 < p, q < 1$ such that the capacities of BSC_p and BEC_q are equal. Let $\mathcal{C}$ be a binary code with good performance on the BEC_q. What can be said about the performance of $\mathcal{C}$ on the BSC_p?

In principle, one could hope that a code with good performance on one channel will also perform well on another channel with the same capacity. However, this is not always the case. Still, our new notion allows to draw nontrivial conclusions about list-decoding on the BSC.

Theorem 3: Let $0 < p < \frac{1}{2}, 0 < q < 1$ and $\eta \geq 0$ such that $\mathrm{BSC}_p + \eta \succeq_{\mathrm{mc}} \mathrm{BEC}_q$. Let $\mathcal{C} = \{\mathcal{C}_n\}$ be a linear binary transitive code, let $X \in \mathbb{F}_2^n$ be uniform over $\mathcal{C}_n$ and $Y_{\mathrm{BEC}} = \mathrm{BEC}_q(X)$ and $Y_{\mathrm{BSC}} = \mathrm{BSC}_p(X)$.
If $H(X|Y_{\mathrm{BEC}}) = o(n)$, then, for every $\varepsilon > 0$, there exists a list decoder $D : \mathbb{F}_2^n \to \mathcal{C}_n^L$ with $L \leq 2^{(\eta+\varepsilon)n}$ such that:

$$\lim_{n\to\infty} \sup_{x\in\mathcal{C}_n} \Pr[X \notin D(Y_{\mathrm{BSC}}) \mid X = x] = 0\,.$$





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Definition and Basic Properties

Definition: Channel $W : \mathcal{X} \to \mathcal{Y}$ is said to be more capable than $\widetilde{W} : \mathcal{X} \to \widetilde{\mathcal{Y}}$ (written as $W \succeq_{\mathrm{mc}} \widetilde{W}$) if for every X, Y, Z such that $P_{Y|X} = W$ and $P_{Z|X} = \widetilde{W}$ it holds $I(X;Y) \geq I(X;Z)$.
Let $\eta \geq 0$. We will say that W with advantage η is more capable than $\widetilde{W}$ and write $W + \eta \succeq_{\mathrm{mc}} \widetilde{W}$ if for every X, Y, Z as above it holds

$$I(X;Y) + \eta \geq I(X;Z).$$

Proposition: For every $n \geq 1$, if $W + \eta \succeq_{\mathrm{mc}} \widetilde{W}$, then $W^n + \eta n \succeq_{\mathrm{mc}} \widetilde{W}^n$.
Proposition: Let $p, q \in (0,1)$. The following give the η_{mc} values between BEC and BSC:

- $\mathrm{BEC}_q + \max\big(0, q - h(p)\big) \succeq_{\mathrm{mc}} \mathrm{BSC}_p$.
- If $q \leq 4p(1-p)$, then $\mathrm{BSC}_p + h(p) - q \succeq_{\mathrm{mc}} \mathrm{BEC}_q$.
- If $4p(1-p) < q$, then $\mathrm{BSC}_p + [-f(r_0)] \succeq_{\mathrm{mc}} \mathrm{BEC}_q$.

App. 2: Hidden Markov Processes

Let $q, \alpha \in [0, 1/2]$. Let W_i, Z_i and X_1 be independent random variables such that $W_i \sim \mathrm{Ber}(q), Z_i \sim \mathrm{Ber}(\alpha), X_1 \sim \mathrm{Ber}(1/2)$. Then, let $X_{n+1} := X_n \oplus W_{n+1}$ and $Y_n := X_n \oplus Z_n$. The problem is to estimate the *entropy rate*

$$\overline{H}(Y) = \lim_{n\to\infty} \frac{H(Y_1, \ldots, Y_n)}{n}.$$

In the following, given $0 \leq \gamma \leq 1$, let $G \sim \mathrm{Geom}(1-\gamma)$, $\alpha * \beta = \alpha(1-\beta) + (1-\alpha)\beta$ and the k-wise convolution $q^{*k} = \frac{1-(1-2q)^k}{2}$.
Theorem 1:[Ordentlich 2016] For $\gamma = 4\alpha(1-\alpha)$, it holds

$$\overline{H}(Y) \geq h\big(\alpha * h^{-1}(\mathbb{E}\, h(q^{*G}))\big)\,.$$

Theorem 2: In the setting above, it holds:

$$\overline{H}(Y) \geq \sup_{0\leq\gamma\leq 1} \big((1-\gamma)\mathbb{E}h(q^{*G}) + h(\alpha) - \eta(\alpha,\gamma)\big)$$

where $\eta(\alpha, \gamma) = \eta_{\mathrm{mc}}(\mathrm{BSC}_\alpha, \mathrm{BEC}_\gamma)$.