spectral independence, sampling bases of matroids, HDX

Daniel Štefankovič

Kauffman, Oppenheim’18 (High order random walks: beyond spectral gap)

Oppenheim’18 (Local spectral expansion approach to HDX)

Anari, Liu, Oveis Gharan, Vinzant’19 (Log-Concave polynomials I-II-III-IV)

Alev, Lau’20 (Improved analysis of higher order random walks and applications)

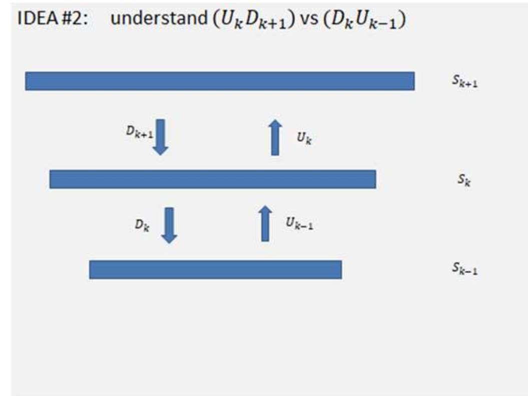
Lau’22 (https://cs.uwaterloo.ca/~lapchi/cs860/notes.html)

Anari, Liu, Oveis Gharan’20 (Spectral independence and HDX and applications to the hardcore model)

Chen, Liu, Vigoda’20 (Rapid Mixing of Glauber Dynamics up to Uniqueness via Contraction)

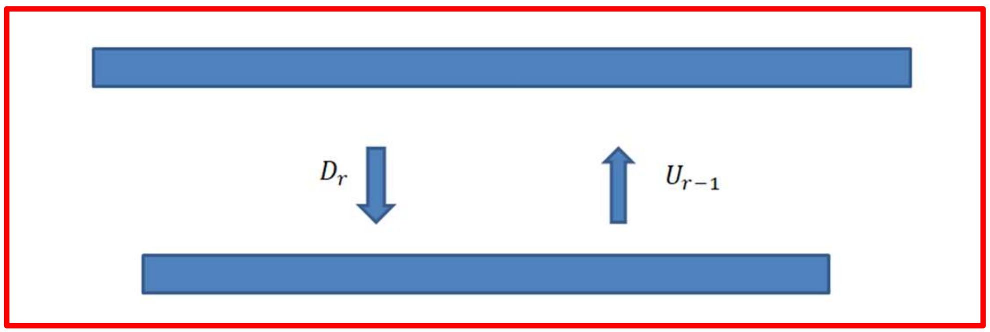
Chen, Galanis, Stefankovic, Vigoda’20 (Rapid Mixing for Colorings via Spectral Independence)

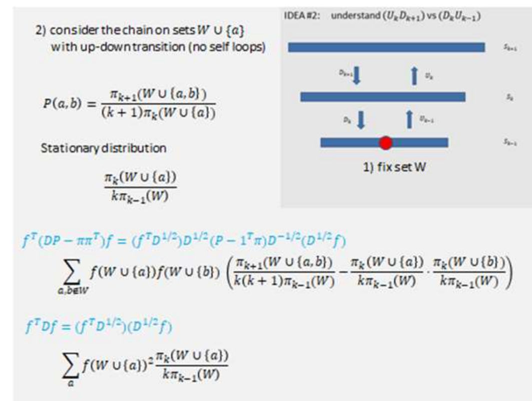
We have shown (Kauffman, Oppenheim'18):



$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I \; - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma I$$

$$I - U_k D_{k+1} \succcurlyeq_{\pi_k} \frac{k}{k+1} (I - D_k U_{k-1}) - \gamma I$$





$$\text{If } \gamma \leq 0 \text{ then } 1 - \lambda_2(D_r U_{r-1}) \geq \frac{1}{r+1}$$

Anari, Liu, Oveis Gharan, Vinzant'19

Can efficiently sample bases of a matroid.

matroids background

fast mixing for matroids

Matroid (independent sets definition) (Ω, I) $\Omega = [n]$

I = independent sets (subsets of $[n]$)

Downward closed

$$T \in \Omega, S \subseteq T \Rightarrow S \in \Omega$$

Independent set exchange property

$$S, T \in I, |S| < |T| \Rightarrow \exists a \in T \setminus S \text{ such that } S \cup \{a\} \in I$$

Matroid (basis definition) (Ω, B) $\Omega = [n]$

B = basis (subsets of $[n]$)

Basis exchange property

$$S, T \in B, S \neq T \Rightarrow \exists a \in T \setminus S, b \in S \setminus T \text{ such that } S \setminus \{b\} \cup \{a\} \in B$$

Matroids

Distribution π_r on all size-r subsets of $[n]$

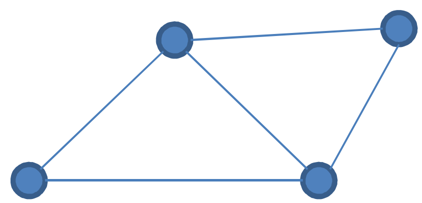
Can we sample from π_r efficiently?

all basis have the same size (rank r)

WANT: sample uniform distribution on basis.

Examples of matroids:

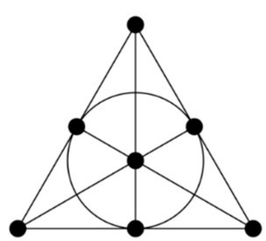
Spanning trees



Independent sets of vectors

$(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,1,1)$

Fano plane



Matroids

Matroid (independent sets definition) (Ω, I) $\Omega = [n]$

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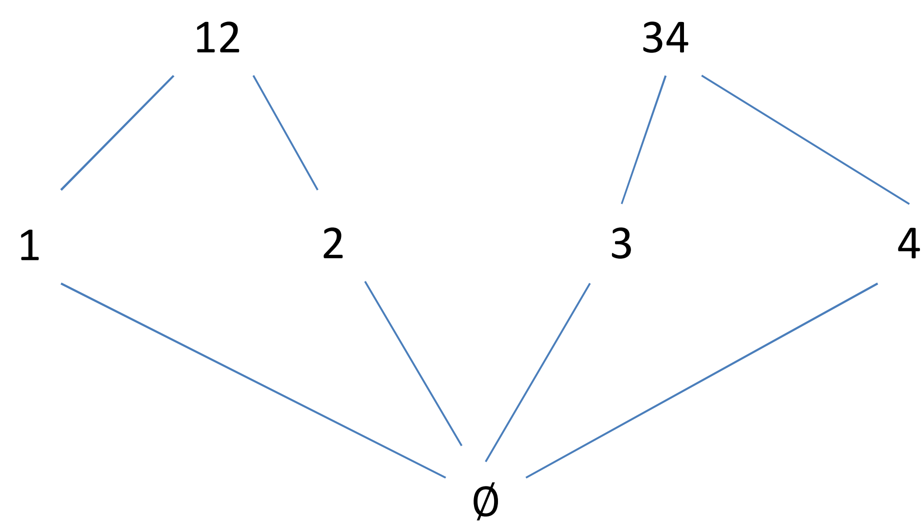
Matroid (basis definition) (Ω, B) $\Omega = [n]$

B = basis (subsets of $[n]$)

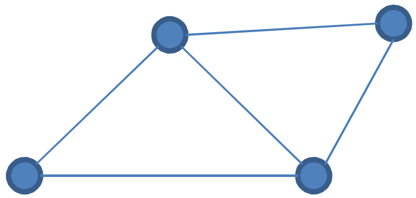
Basis exchange property

$$S, T \in B, S \neq T \Rightarrow \exists a \in T \setminus S, b \in S \setminus T \text{ such that } S \setminus \{b\} \cup \{a\} \in B$$

A ``non-example'' of a matroid:



Spanning trees ("graphic")



Matroid (independent sets definition) (Ω, I) $\Omega = [n]$

I = independent sets (subsets of $[n]$)

Downward closed

$$T \in \Omega, S \subseteq T \Rightarrow S \in \Omega$$

Independent set exchange property

$$S, T \in I, |S| < |T| \Rightarrow \exists a \in T \setminus S \text{ such that } S \cup \{a\} \in I$$

Matroid (basis definition) (Ω, B) $\Omega = [n]$

B = basis (subsets of $[n]$)

Basis exchange property

$$S, T \in B, S \neq T \Rightarrow \exists a \in T \setminus S, b \in S \setminus T \text{ such that } S \setminus \{b\} \cup \{a\} \in B$$

Independent sets of vectors ("representable")

$(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,1,1)$

Matroid (independent sets definition) (Ω, I) $\Omega = [n]$

I = independent sets (subsets of $[n]$)

Downward closed

$T \in \Omega, S \subseteq T \Rightarrow S \in \Omega$

Independent set exchange property

$S, T \in I, |S| < |T| \Rightarrow \exists a \in T \setminus S$ such that $S \cup \{a\} \in I$

Matroid (basis definition) (Ω, B) $\Omega = [n]$

B = basis (subsets of $[n]$)

Basis exchange property

$S, T \in B, S \neq T \Rightarrow \exists a \in T \setminus S, b \in S \setminus T$ such that $S \setminus \{b\} \cup \{a\} \in B$

Matroids

deletion: $M - e$

$$(\Omega \setminus \{e\}, I')$$

$$I' = \{S \mid S \in I \text{ and } e \notin S\}$$

contraction: M/e

$$(\Omega \setminus \{e\}, I')$$

$$I' = \{S \mid S \cup \{e\} \in I\}$$

matroids background

fast mixing for matroids

I = independent sets (subsets of $[n]$)

Independent set exchange property

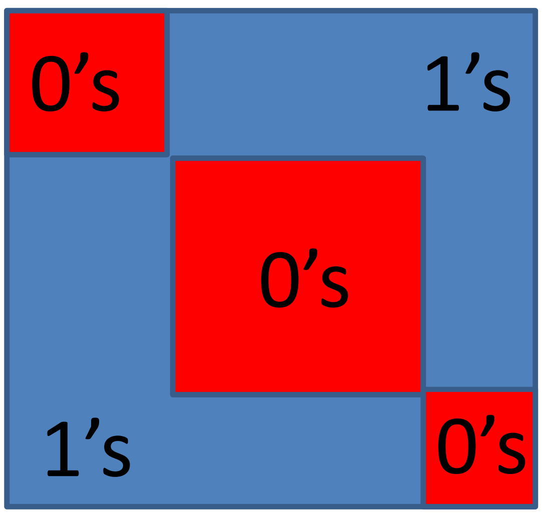
$$S, T \in I, |S| < |T| \Rightarrow \exists a \in T \setminus S \text{ such that } S \cup \{a\} \in I$$

Matroids

How do matroids of rank 2 look like?

$$\{a,c\} \in B \Rightarrow \{a,b\} \in B \ \text{ or } \ \{b,c\} \in B$$

$$\{a,b\} \notin B \ \text{ and } \ \{b,c\} \notin B \ \Rightarrow \{a,c\} \notin B$$



Kauffman, Oppenheim’18

$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma I$$

$$I - U_k D_{k+1} \succcurlyeq_{\pi_k} \frac{k}{k+1}(I - D_k U_{k-1}) - \gamma I$$

If $\gamma \leq 0$ then $1 - \lambda_2(D_r U_{r-1}) \geq \frac{1}{r+1}$

Anari, Liu, OveisGharan, Vinzant’19

Can efficiently sample bases of a matroid.

Matroid (independent sets definition) (Ω, I) $\Omega = [n]$

I = independent sets (subsets of $[n]$)

Downward closed

$T \in \Omega, S \subseteq T \Rightarrow S \in \Omega$

Independent set exchange property

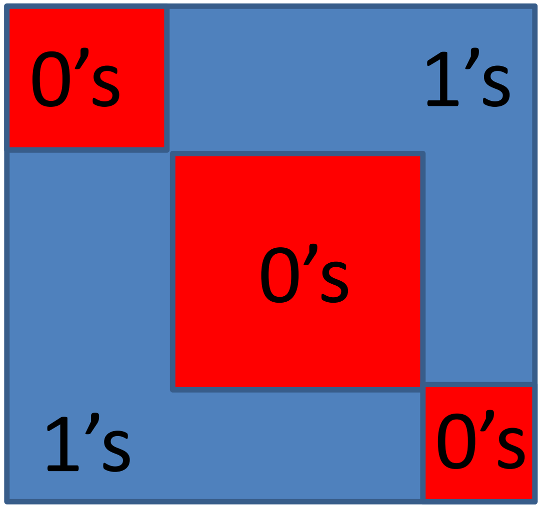
$S, T \in I, |S| < |T| \Rightarrow \exists a \in T \setminus S$ such that $S \cup \{a\} \in I$

Matroid (basis definition) (Ω, B) $\Omega = [n]$

B = basis (subsets of $[n]$)

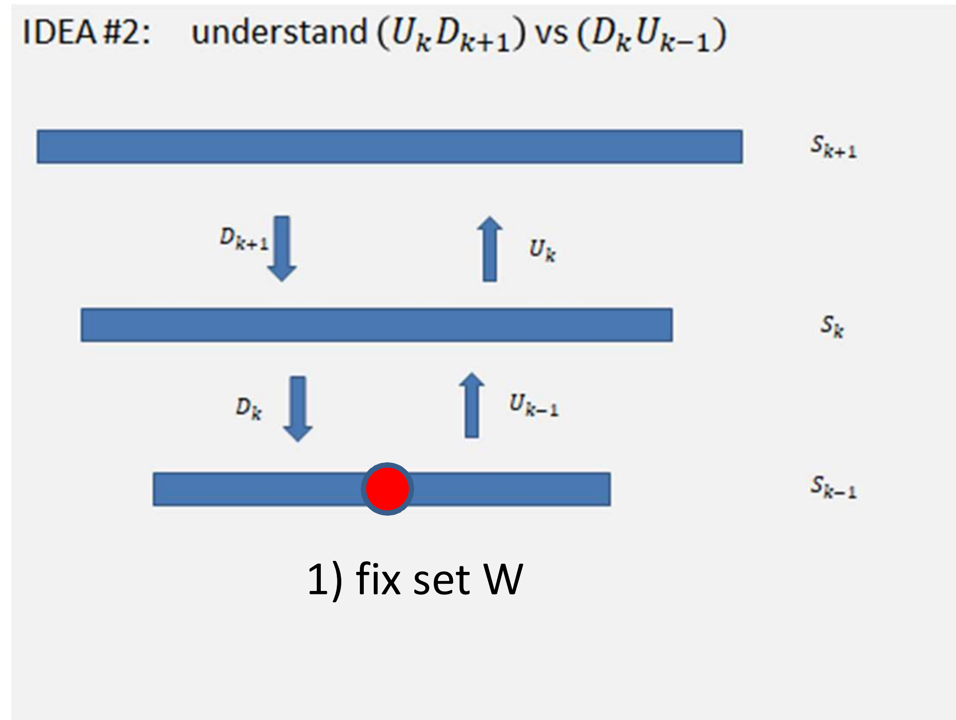
Basis exchange property

$S, T \in B, S \neq T \Rightarrow \exists a \in T \setminus S, b \in S \setminus T$ such that $S \setminus \{b\} \cup \{a\} \in B$



$$\gamma \leq 0$$

(for matroids of rank 2)

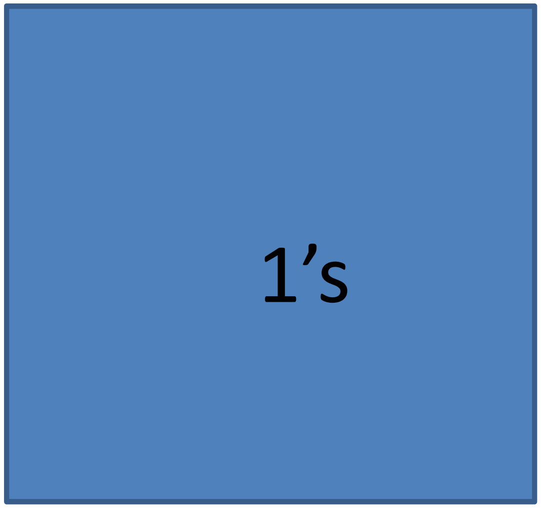


consider the chain on sets $W \cup \{a\}$
with up-down transition (no self loops)

$$P(a,b) = \frac{\pi_{k+1}(W \cup \{a,b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

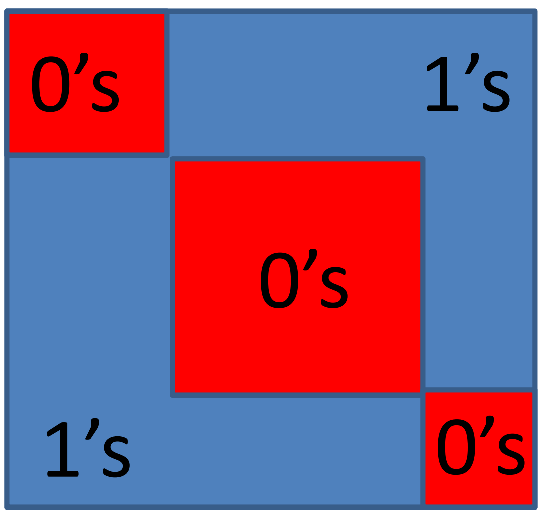


$= J$

$0, \ldots, 0, n$

$A \preceq 0$

$$\min_{\substack{\text{subspaces} \\ \text{of dimension } n-k}} \ \max_{x \in S} \frac{x^T J x}{\|x\|_2}$$



$A \succeq 0$

$= J - A$

eigenvalues cannot increase

$$\min_{\substack{\text{subspaces} \\ \text{of dimension} \\ n-k}} \ \max_{x \in S} \frac{x^T J x - x^T A x}{\|x\|_2^2}$$

Rank 3?

π_3 uniform

WANT: bound on λ_2 for the up-down (no self-loops) chain on X(1)

HAVE: bound on λ_2 (≤ 0) for the up-down (no self-loops) chain on X(2)
when restricted to sets containing element a

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2}) D^{1/2}(P - 1^T\pi) D^{-1/2}(D^{1/2} f)$$

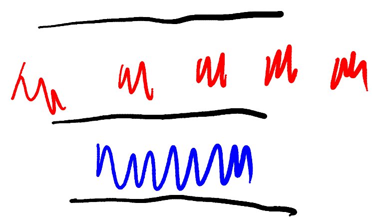
$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

WANT: bound on λ_2 for the up-down chain on X(1)

$$\pi_k(S) = \binom{r}{k}^{-1} \sum_{\substack{T;S\subseteq T\\ |T| = r}} \pi_r(T)$$

$$(DP)_{ik} = \pi_2(\{i,k\})/2$$

$$(D)_{ii} = \pi_1(\{i\})$$



HAVE: bound on λ_2 (≤ 0) for the up-down chain on X(2)
when restricted to sets containing element a

$$\pi_a(i) = \frac{\pi_2(\{a,i\})}{\pi_1(\{a\})}$$

$$(D_a P_a)_{ik} = \frac{1}{\pi_1(\{a\})}\pi_3(\{a,i,k\})/6$$

$$(D_a)_{ii} = \frac{1}{\pi_1(\{a\})}\pi_2(\{a,i\})/2$$

$$DP = \sum_a \pi_1(\{a\}) D_a P_a$$

$$D = \sum_a \pi_1(\{a\}) D_a$$

$$\pi = \sum_a \pi_1(\{a\}) \pi_a$$

For f such that $\sum_i \pi_2(\{i,a\}) f(i) = 0$

$$f^T(D_a P_a) f \leq \gamma f^T D_a f$$

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2}) D^{1/2}(P - 1^T\pi) D^{-1/2}(D^{1/2} f)$$

$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

WANT: bound on λ_2 for the up-down chain on X(1)

$$\pi_k(S) = \binom{r}{k}^{-1} \sum_{\substack{T; S \subseteq T \\ |T| = r}} \pi_r(T)$$

$$(DP)_{ik} = \pi_2(\{i,k\})/2$$

$$(D)_{ii} = \pi_1(\{i\})$$

HAVE: bound on λ_2 (≤ 0) for the up-down chain on X(2) when restricted to sets containing element a

$$\pi_a(i) = \frac{\pi_2(\{a,i\})}{\pi_1(\{a\})}$$

$$(D_a P_a)_{ik} = \frac{1}{\pi_1(\{a\})} \pi_3(\{a,i,k\})/6$$

$$(D_a)_{ii} = \frac{1}{\pi_1(\{a\})} \pi_2(\{a,i\})/2$$

$$DP = \sum_a \pi_1(\{a\}) D_a P_a$$

$$D = \sum_a \pi_1(\{a\}) D_a$$

For f such that $\sum_i \pi_2(\{i,a\}) f(i) = 0$

$$f^T(D_a P_a) f \leq \gamma f^T D_a f$$

$$\pi = \sum_a \pi_1(\{a\}) \pi_a$$

f eigenvector of DP under D for λ_2

$$\pi^T f = 0$$

$$DP\, f = \lambda_2 D\, f$$

$$\lambda_2 f^T D f = f^T\, DP\, f = f^T \left(\sum_a \pi_1(\{a\}) (D_a P_a - \pi_a \pi_a^T + \pi_a \pi_a^T) \right) f \leq$$

$$\sum_a \pi_1(\{a\})\ (\gamma\, f^T (D_a - \pi_a \pi_a^T)\, f + f^T \pi_a \pi_a^T f) =$$

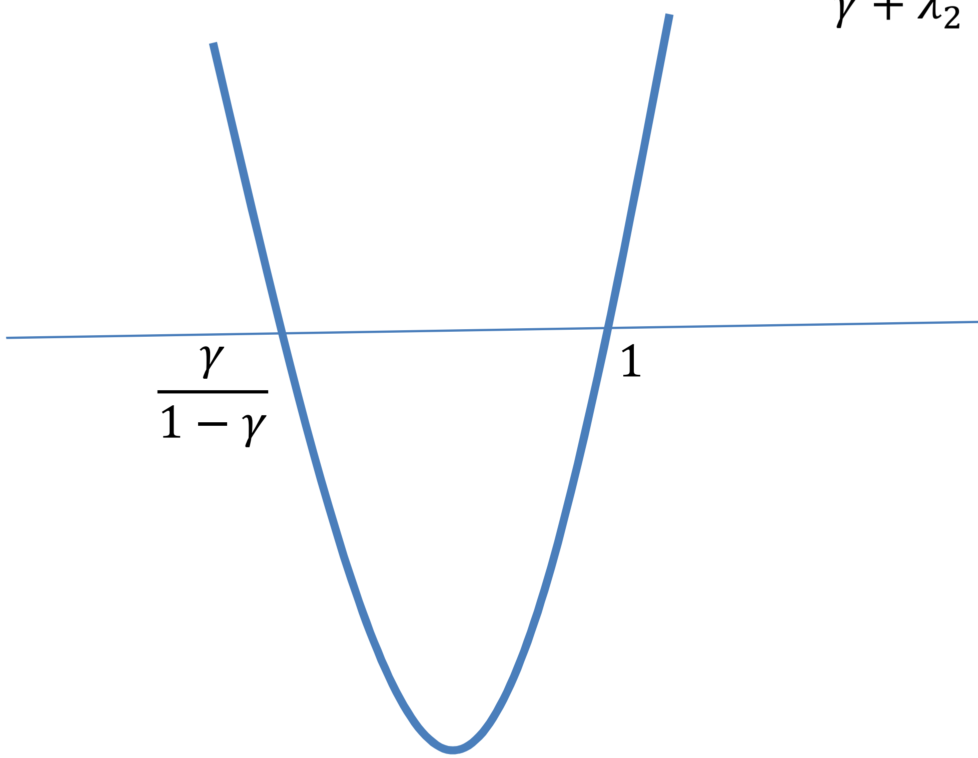
$$\gamma f^T D f + (1-\gamma) \sum_a \pi_1(\{a\}) (f^T \pi_a)^2 = \gamma f^T D f + (1-\gamma) \lambda_2^2 f^T D f$$

$$f^T \pi_a = \frac{1}{\pi_1(\{a\})} (f^T DP)[a] = \lambda_2 \frac{1}{\pi_1(\{a\})} (f^T D)[a] = \lambda_2 f[a]$$

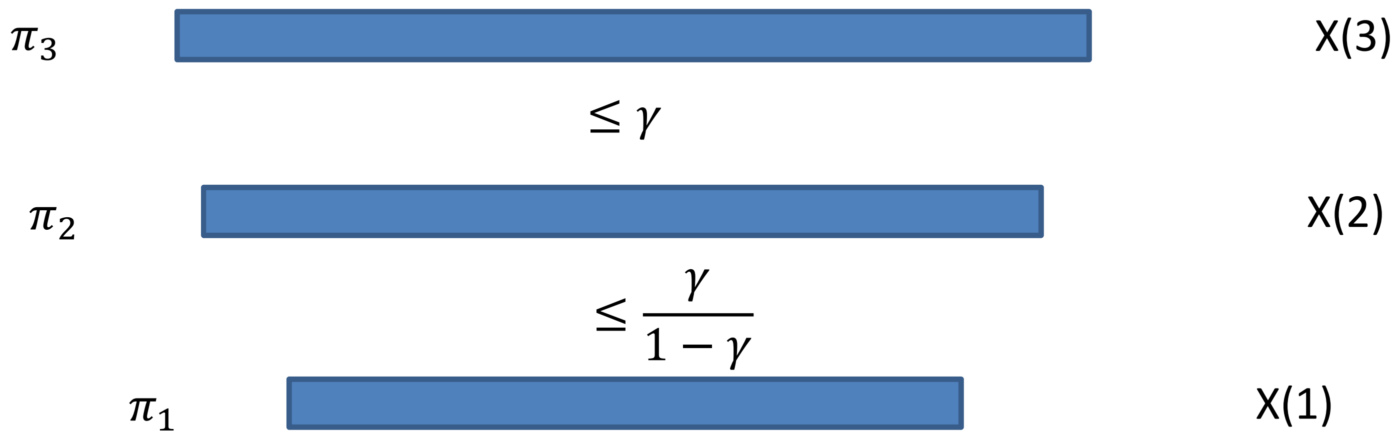
$$\lambda_2 f^T D f \leq (\gamma + (1-\gamma)\lambda_2^2) f^T D f$$

$$\lambda_2 \leq \gamma + (1-\gamma)\lambda_2^2$$

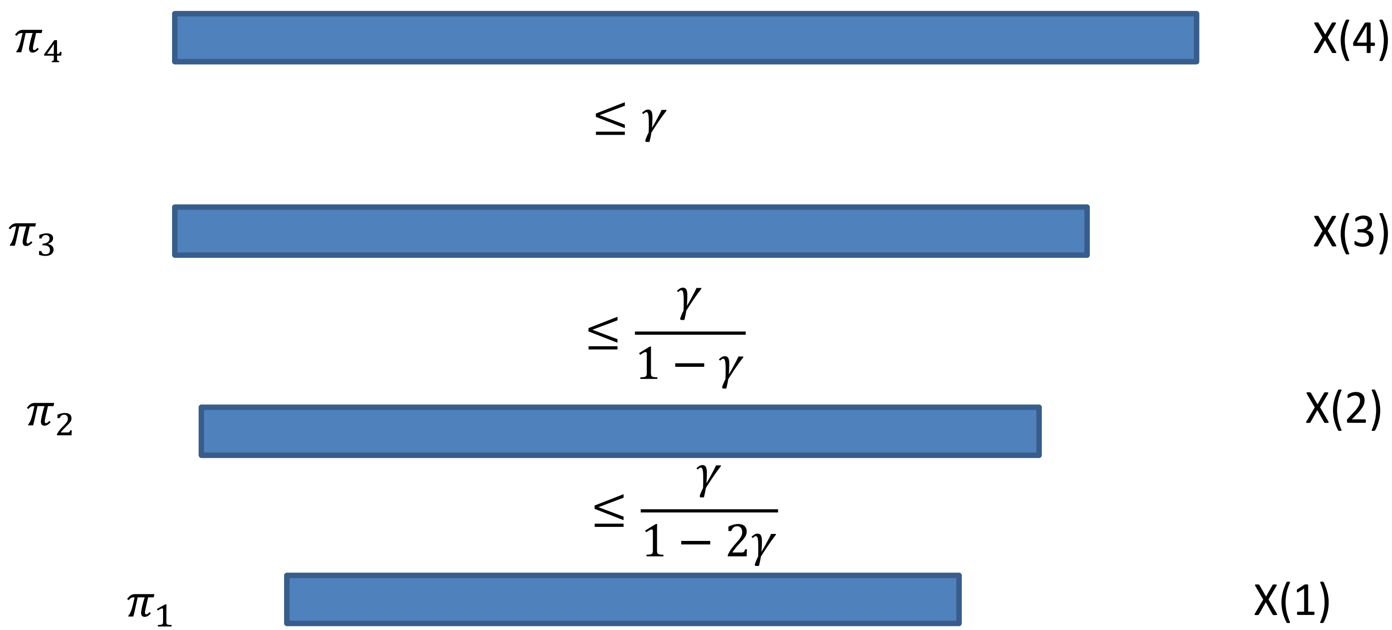
$\gamma + \lambda_2 + (1-\gamma)\lambda_2^2$



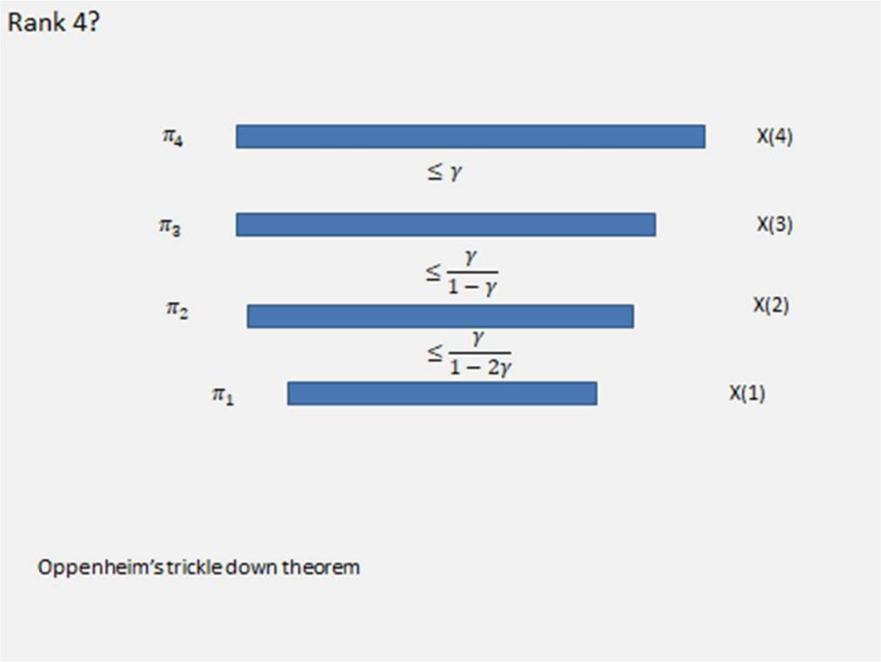
Rank 3?

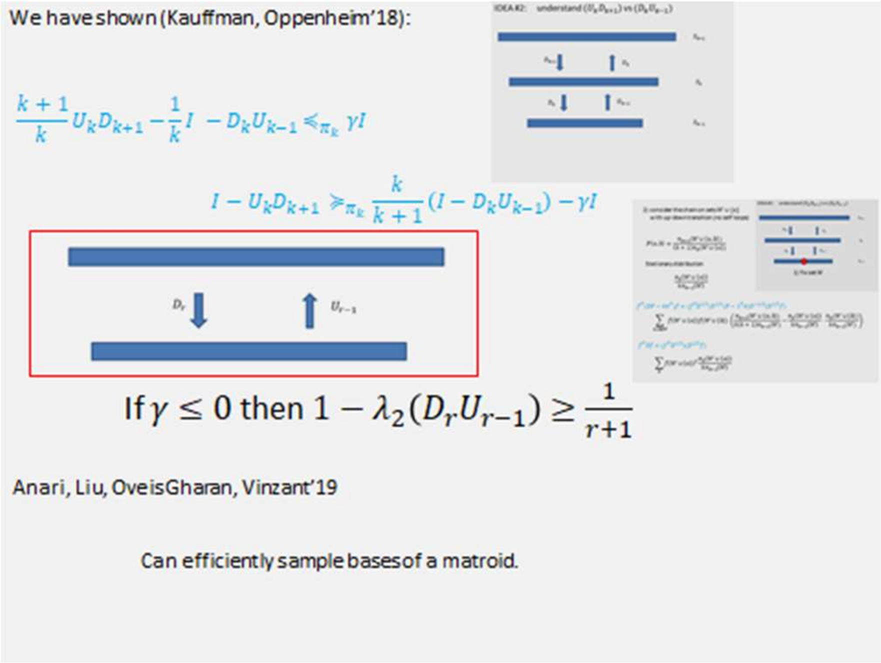


Rank 4?



Oppenheim’s trickle down theorem





$$\gamma \leq 0$$

for all local chains

Strongly log-concave distributions

$$\frac{f\,(\nabla^2 f) - (\nabla f)(\nabla f)^{\mathrm{T}}}{f^2}$$

distribution π_r on subsets of size r of $[n]$ ⇒ homogeneous polynomial

$$f = \sum_{S\subseteq[n],|S|=r} \pi_S \prod_{i\in S} x_i$$

Definition: π_r is log-concave ⇔ $\log f$ is concave at 1

Definition: π_r is strongly log-concave ⇔ For any $x_1, \ldots, x_k$
$\log(\partial x_{i_1} \ldots \partial x_{i_k} f)$
Is concave at 1

Lemma: π_r is strongly log-concave if and only if all local walks have $\lambda_2 \le 0$

Corollary: uniform distribution on bases of a matroid is strongly log-concave

Lorentzian polynomials

Petter Brändén, June Huh

We study the class of Lorentzian polynomials. The class contains homogeneous stable polynomials as well as volume polynomials of convex bodies and projective varieties. We prove that the Hessian of a nonzero Lorentzian polynomial has exactly one positive eigenvalue at any point on the positive orthant. This property can be seen as an analog of Hodge--Riemann relations for Lorentzian polynomials. Lorentzian polynomials are intimately connected to matroid theory and negative dependence properties. We show that matroids, and more generally M-convex sets, are characterized by the Lorentzian property, and develop a theory around Lorentzian polynomials. In particular, we provide a large class of linear operators that preserve the Lorentzian property and prove that Lorentzian measures enjoy several negative dependence properties. We also prove that the class of tropicalized Lorentzian polynomials coincides with the class of M-convex functions in the sense of discrete convex analysis. The tropical connection is used to produce Lorentzian polynomials from M-convex functions. We give two applications of the general theory. First, we prove that the homogenized multivariate Tutte polynomial of a matroid is Lorentzian whenever the parameter q satisfies $0 < q \leq 1$. Consequences are proofs of the strongest Mason's conjecture from 1972 and negative dependence properties of the random cluster model model in statistical physics. Second, we prove that the multivariate characteristic polynomial of an M-matrix is Lorentzian. This refines a result of Holtz who proved that the coefficients of the characteristic polynomial of an M-matrix form an ultra log-concave sequence.

Comments: 60 pages, with a new remark on Question 4.9
Subjects: **Combinatorics (math.CO)**; Algebraic Geometry (math.AG); Probability (math.PR)
Cite as: arXiv:1902.03719 **[math.CO]**
(or arXiv:1902.03719v5 **[math.CO]** for this version)
https://doi.org/10.48550/arXiv.1902.03719

Hodge Theory for Combinatorial Geometries

Karim Adiprasito, June Huh, Eric Katz

We prove the hard Lefschetz theorem and the Hodge-Riemann relations for a commutative ring associated to an arbitrary matroid M. We use the Hodge-Riemann relations to resolve a conjecture of Heron, Rota, and Welsh that postulates the log-concavity of the coefficients of the characteristic polynomial of M. We furthermore conclude that the f-vector of the independence complex of a matroid forms a log-concave sequence, proving a conjecture of Mason and Welsh for general matroids.

Comments: 63 pages. Minor revision
Subjects: **Combinatorics (math.CO)**; Algebraic Geometry (math.AG)
Cite as: arXiv:1511.02888 **[math.CO]**
(or arXiv:1511.02888v2 **[math.CO]** for this version)
https://doi.org/10.48550/arXiv.1511.02888

Spectral independence

Alev, Lau'20

$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I \ - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma (I - D_k U_{k-1})$$

$$I - U_k D_{k+1} \ \succcurlyeq_{\pi_k} (1-\gamma) \frac{k}{k+1} (I - D_k U_{k-1})$$

$$\text{If } \gamma_k \leq \frac{C}{\underset{r}{\cancel{n}} - k} \quad \text{then } 1 - \lambda_2 (D_r U_{r-1}) \geq r^{-O(C)}$$

Anari, Liu, Oveis Gharan'20

Can efficiently sample from antiferromagnetic 2-spin models in uniqueness.

hard-core model

colorings

encoding as simplicial complex/hypergraph

spectral independence

weak spatial mixing, strong spatial mixing

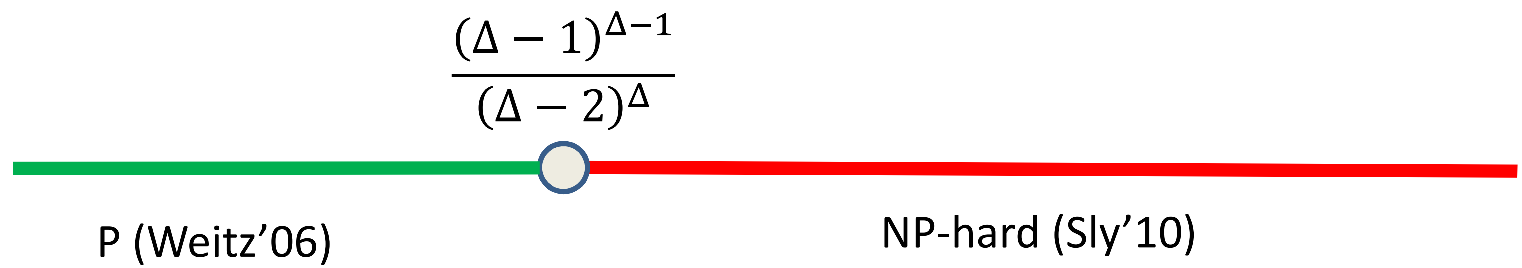
Hard-core model:

Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $\lambda > 0$

distribution on independent sets of G

$$P(S) \propto \lambda^{|S|}$$

Can we efficiently sample from the distribution?



hard-core model

colorings

encoding as simplicial complex/hypergraph

spectral independence

weak spatial mixing, strong spatial mixing

Colorings:

Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $q \in \mathbb{N}$

uniform distribution on q-coloring of G

Can we efficiently sample from the distribution?

$\frac{\Delta}{\log \Delta}$ $\quad \Delta - \sqrt{\Delta}$ $\quad \Delta$ $\quad \Delta+1$ $\quad \left(\frac{11}{6} - \varepsilon\right)\Delta$

NP-hard (Galanis, Stefankovic, Vigoda'14)

(even for triangle-free graphs)

Chen, Delcourt, Moitra, Perarnau, Postle'19
Vigoda'99
Jerrum'95

hard-core model

colorings

encoding as simplicial complex/hypergraph

spectral independence

weak spatial mixing, strong spatial mixing

Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $\lambda > 0$

distribution on independent sets of G

$$P(S) \propto \lambda^{|S|}$$

$$\Omega = \{ (v, \sigma) \mid v \in V, \sigma \in \{0,1\}\}$$

- only allow subsets of Ω that are valid
- on the top-level

$$P(A) = P(S)$$

where S is the corresponding independent set

Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $\lambda > 0$

distribution on independent sets of G

$$P(S) \propto \lambda^{|S|}$$

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- only allow subsets of Ω that are valid
- on the top-level

$$P(A) = P(S)$$

where S is the corresponding independent set

$$\Omega = \{ (A, \emptyset), (A, 1), \ldots, (D, \emptyset), (D, 1) \}$$

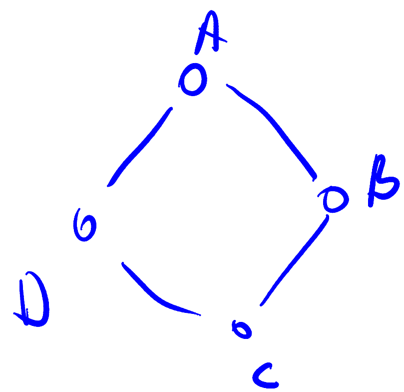
$$r = 4$$

$$\{ (A, \emptyset), (B, \emptyset), (C, \emptyset), (D, \emptyset) \}$$

$$P(\cdot) = \frac{1}{1 + 4\lambda + \lambda^2}$$

$$\{ (A, 1), (B, \emptyset), (C, 1), (D, \emptyset) \}$$

$$P(\cdot) = \frac{\lambda^2}{1 + 4\lambda + \lambda^2}$$



Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $q \in \mathbb{N}$

uniform distribution on q-colorings of G

$$\Omega = \{ (v, \sigma) \mid v \in V, \sigma \in \{1,2, \ldots, q\} \}$$

- only allow subsets of Ω that are valid
- on the top-level

$$P(A) = P(\tau)$$

where τ is the corresponding coloring

hard-core model

colorings

encoding as simplicial complex/hypergraph

spectral independence

weak spatial mixing, strong spatial mixing

``Local'' chains

$$P(a,b) = \frac{\pi_{k+1}(W \cup \{a,b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $\lambda > 0$

distribution on independent sets of G

$$P(S) \propto \lambda^{|S|}$$

$$\Omega = \{\, (v, \sigma) \mid v \in V, \sigma \in \{0,1\}\}$$

- only allow subsets of Ω that are valid
- on the top-level

$$P(A) = P(S)$$

where S is the corresponding independent set

``Local'' chains (with notation matching the pairs)

$$P((u,a),(v,b)) = \frac{\pi_{k+1}(W \cup \{(u,a),(v,b)\})}{(k+1)\pi_k(W \cup \{(u,a)\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{(u,a)\})}{k\pi_{k-1}(W)}$$

What is $\pi_k(\ \{\ (v_1, a_1), \dots, (v_k, a_k)\ \})$?

$|W| = k - 1$

$\binom{n}{k}^{-1}$ probability that a random (from the measure of the model) assignment satisfies $\tau(v_i) = a_i$

$$P((u,a),(v,b)) = \frac{\pi_{k+1}(W \cup \{(u,a),(v,b)\})}{(k+1)\pi_k(W \cup \{(u,a)\})} = \frac{\binom{n}{k+1}^{-1} P(\ \tau(u) = a \wedge \tau(v) = b \wedge W)}{(k+1)\binom{n}{k}^{-1} P(\ \tau(u) = a \wedge W)}$$

$$= \frac{1}{n-k} \frac{P(\ \tau(u) = a \wedge \tau(v) = b \wedge W\ )}{P(\ \tau(u) = a \wedge W)}$$

$$\pi(u,a) = \frac{\pi_k(W \cup \{(u,a)\})}{k\pi_{k-1}(W)} = \frac{\binom{n}{k}^{-1} P(\ \tau(u) = a \wedge \tau(v) = b \wedge W\ )}{k\binom{n}{k-1}^{-1} P(\ \tau(u) = a \wedge W)}$$

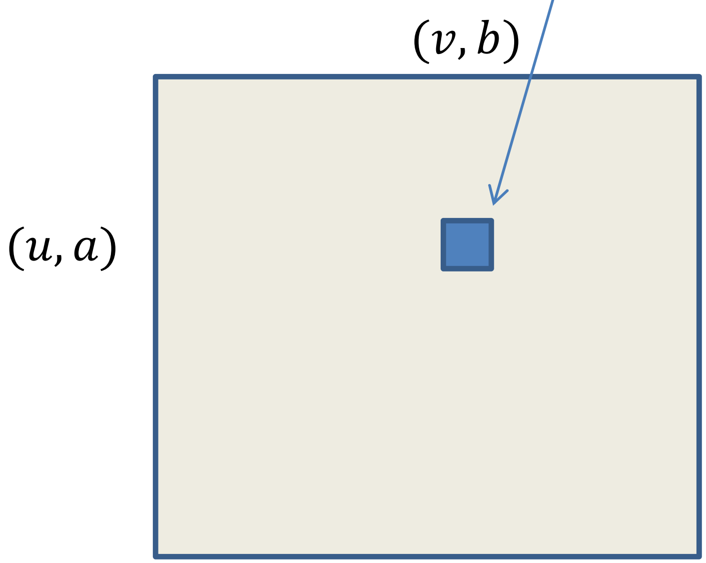
$$= \frac{1}{n-k+1} \frac{P(\ \tau(u) = a \wedge W\ )}{P(\ W)}$$

For every partial assignment W

$$P(\tau(v) = b \mid \tau(u) = a) - P(\tau(v) = b)$$

(zero if $u = v$)

Influence matrix



If $\lambda_2(\mathrm{M}) = O(1)$ then $\lambda_2(\text{local chains}) \leq \frac{C}{n-k}$

What is $\pi_k(\{(v_1, a_1), \ldots, (v_k, a_k)\})$? $|W| = k-1$

$\binom{n}{k}^{-1}$ probability that a random (from the measure of the model) assignment satisfies $\tau(v_i) = a_i$

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$$= \frac{1}{n-k} \frac{P(\tau(u) = a \wedge \tau(v) = b \wedge W)}{P(\tau(u) = a \wedge W)}$$

$$\pi(u,a) = \frac{\pi_k(W \cup \{(u,a)\})}{k\pi_{k-1}(W)} = \frac{\binom{n}{k}^{-1} P(\tau(u) = a \wedge \tau(v) = b \wedge W)}{k\binom{n}{k-1}^{-1} P(\tau(u) = a \wedge W)}$$

$$= \frac{1}{n-k+1} \frac{P(\tau(u) = a \wedge W)}{P(W)}$$

Alev, Lau'20

$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma (I - D_k U_{k-1})$$

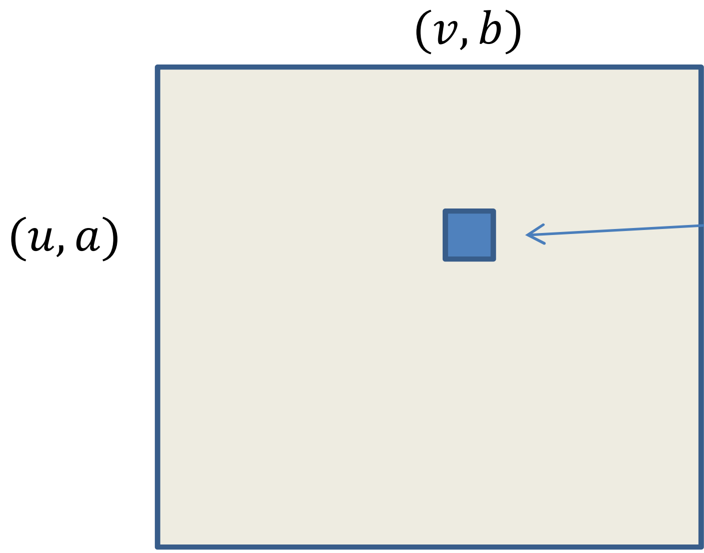
$$I - U_k D_{k+1} \succcurlyeq_{\pi_k} (1-\gamma) \frac{k}{k+1} (I - D_k U_{k-1})$$

If $\gamma_k \leq \frac{C}{n-k}$ then $1 - \lambda_2(D_r U_{r-1}) \geq r^{-O(C)}$

Anari, Liu, OveisGharan'20

Can efficiently sample from antiferromagnetic 2-spin models in uniqueness

Influence matrix



If $\lambda_2(\mathrm{M}) = O(1)$ then $\lambda_2(\text{local chains}) \leq \frac{C}{n-k}$

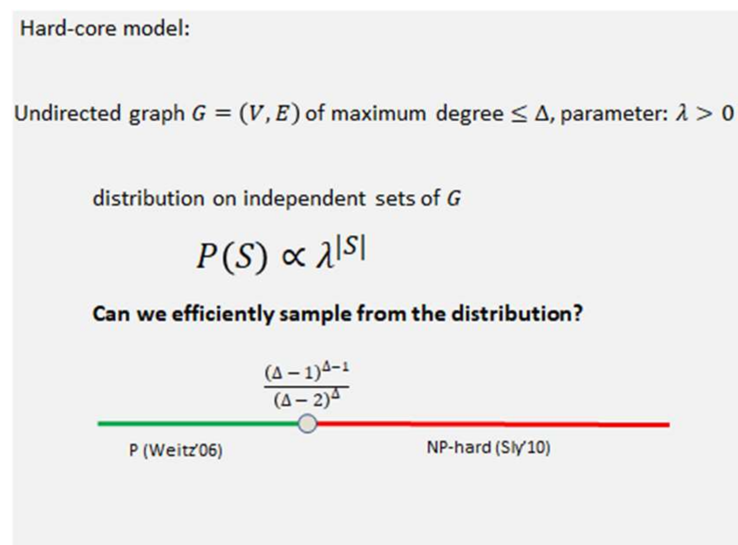
$$P(\tau(v) = b \mid \tau(u) = a) - P(\tau(v) = b)$$

$$\sum_{(v,b)} |P(\tau(v) = b \mid \tau(u) = a) - P(\tau(v) = b)|$$

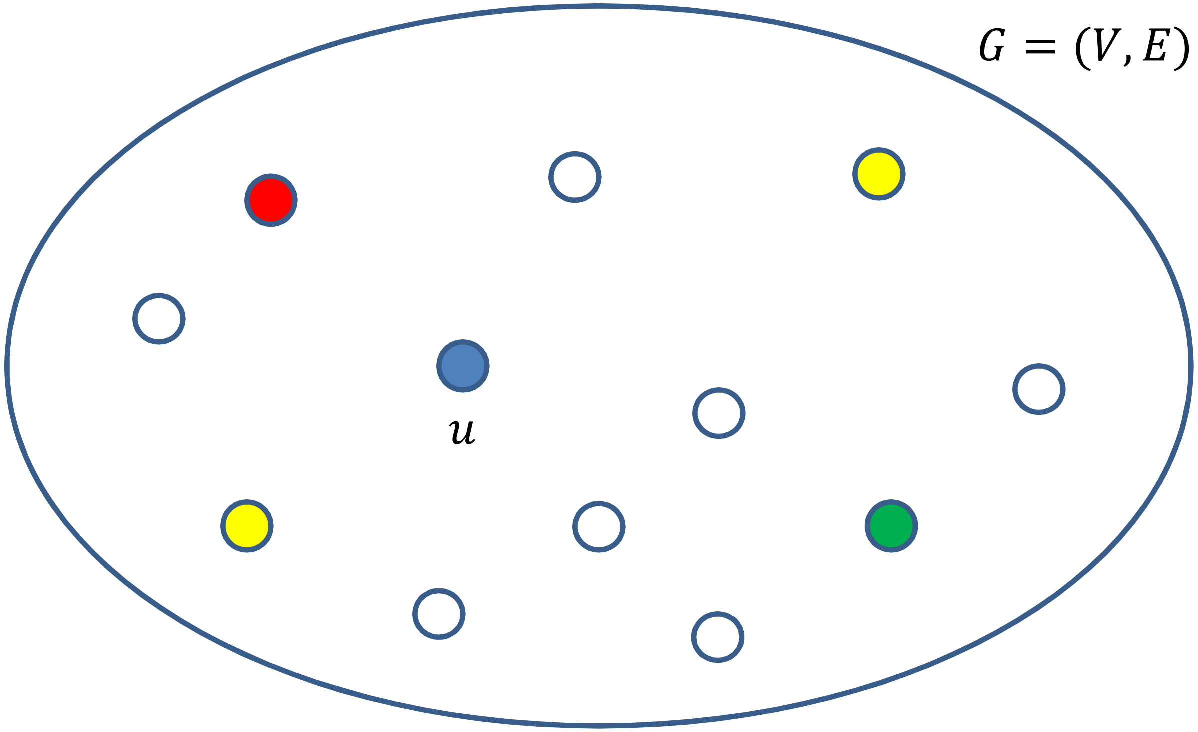
Way to bound $\rho(A)$:

max row norm

$$\max_i \sum_j |A_{ij}|$$



Spectral independence



$$\sum_{(v,b)} |P(\tau(v) = b \mid \tau(u) = a) - P(\tau(v) = b)|$$

hard-core model

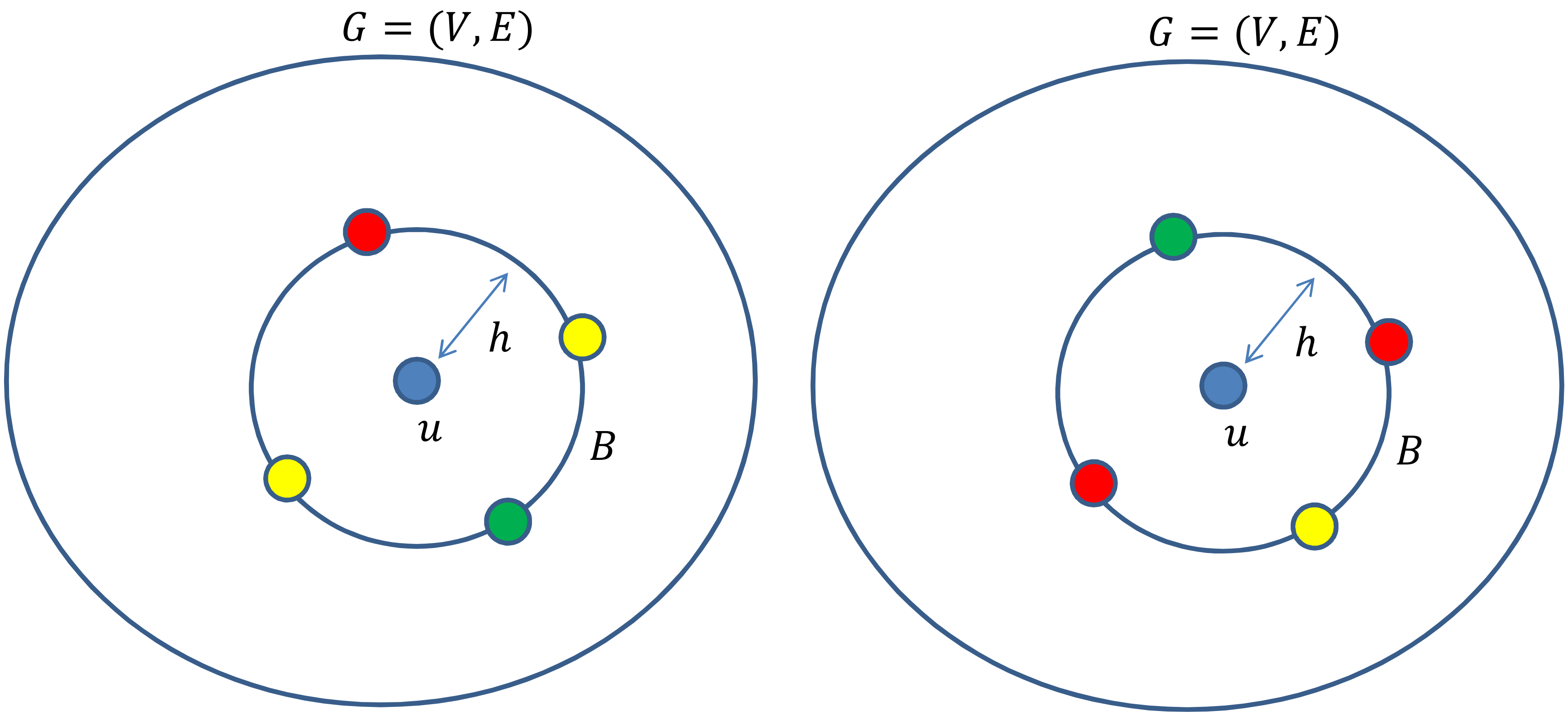
colorings

encoding as simplicial complex/hypergraph

spectral independence

weak spatial mixing, strong spatial mixing

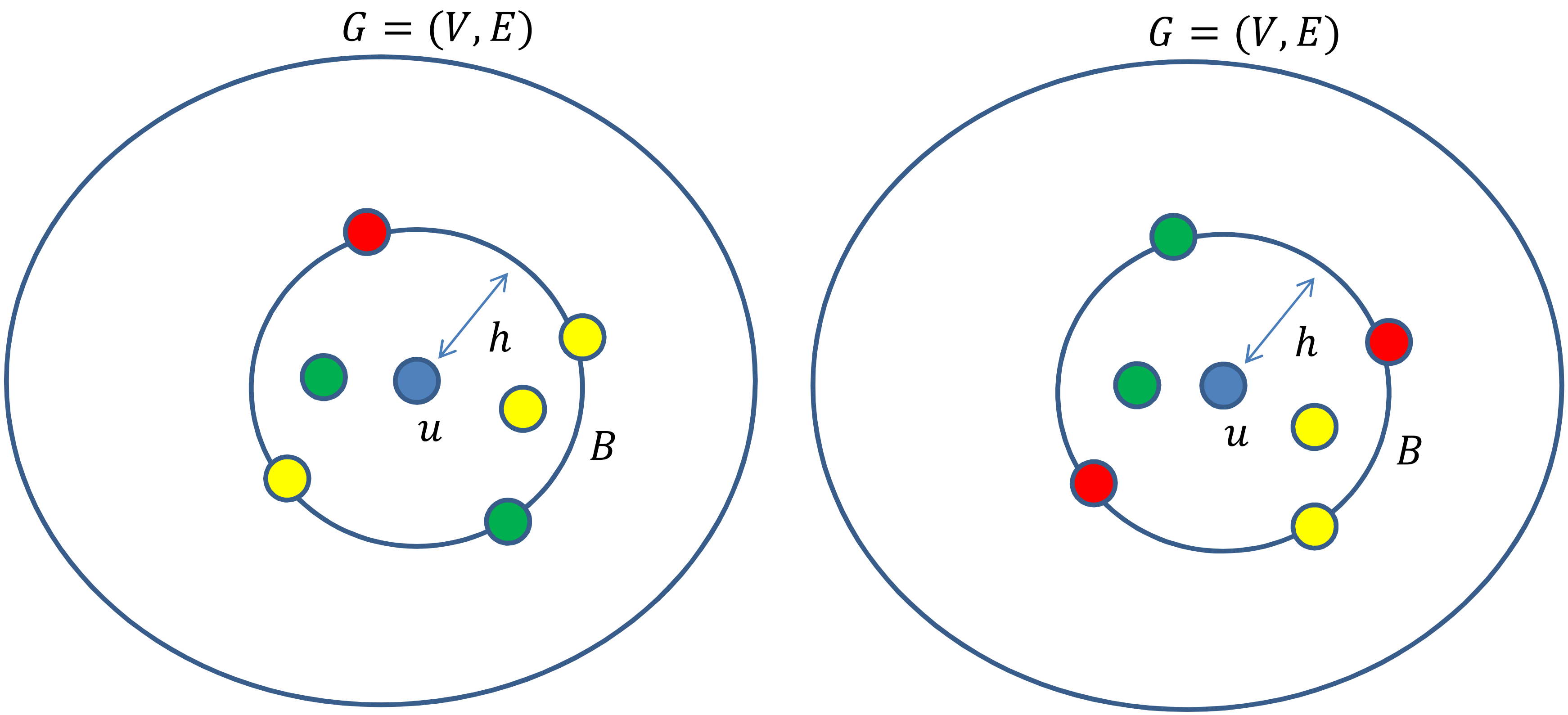
Weak spatial mixing



$$\max|\, P(\sigma(u) = a)\ |\sigma(B) = b_1) - P(\sigma(u) = a|\sigma(B) = b_2)\,| \leq F(dist(u, B))$$

$$\lim_{h\to\infty} F(t) = 0$$

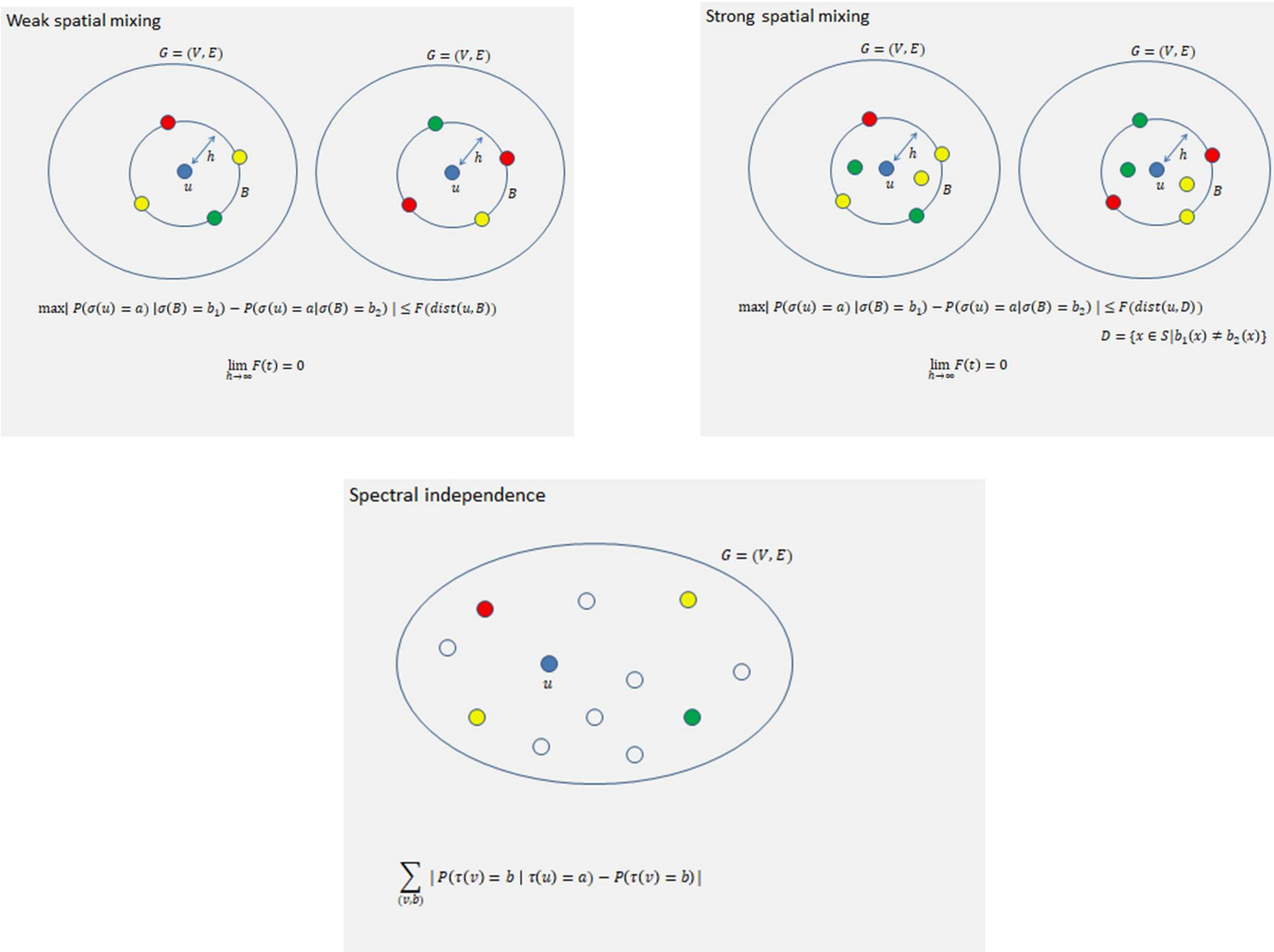
Strong spatial mixing

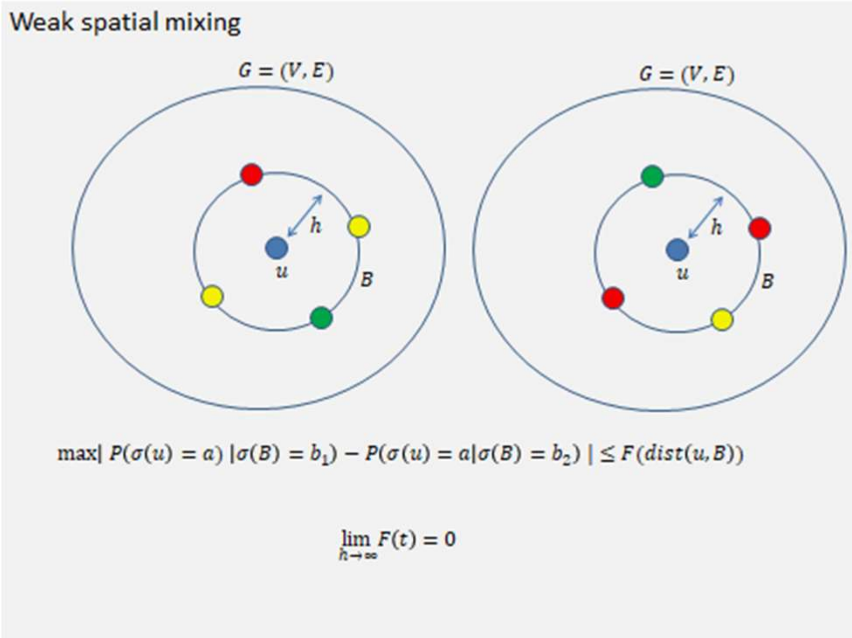


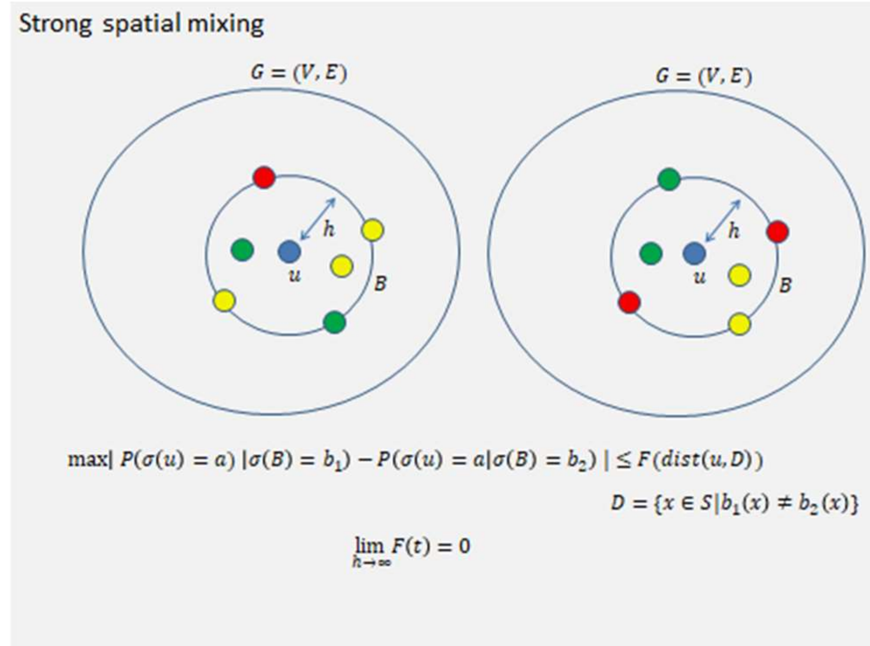
$$\max|\ P(\sigma(u) = a)\ |\sigma(B) = b_1) - P(\sigma(u) = a|\sigma(B) = b_2)\ | \leq F(dist(u, D))$$

$$D = \{x \in S | b_1(x) \neq b_2(x)\}$$

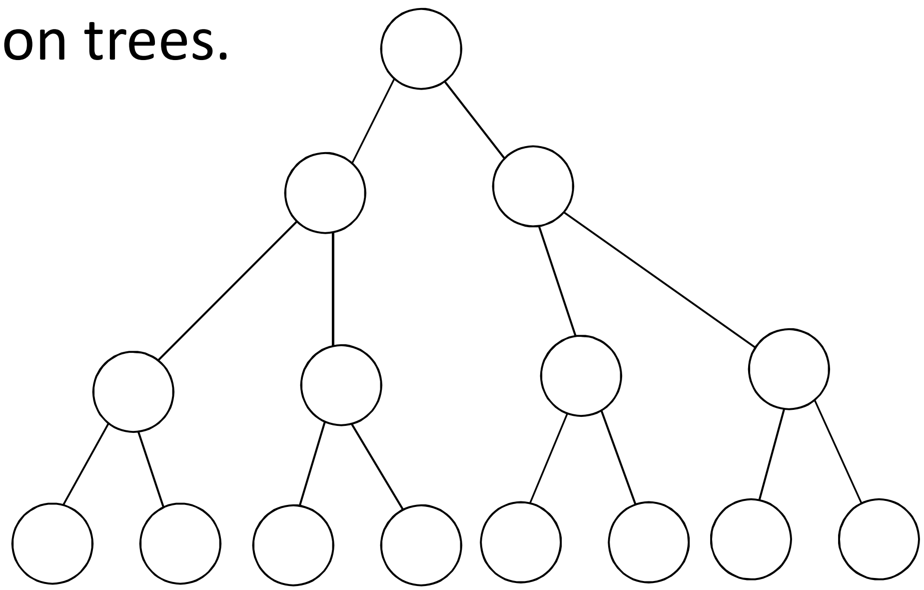
$$\lim_{h \to \infty} F(t) = 0$$


Weak spatial mixing
$G = (V, E)$
$G = (V, E)$
h
u
B
h
u
B
$\max | P(\sigma(u) = a) \, | \sigma(B) = b_1) - P(\sigma(u) = a | \sigma(B) = b_2) \, | \leq F(dist(u, B))$
$\lim_{h \to \infty} F(t) = 0$
Strong spatial mixing
$G = (V, E)$
$G = (V, E)$
h
u
B
h
u
B
$\max | P(\sigma(u) = a) \, | \sigma(B) = b_1) - P(\sigma(u) = a | \sigma(B) = b_2) \, | \leq F(dist(u, D))$
$D = \{x \in S | b_1(x) \neq b_2(x)\}$
$\lim_{h \to \infty} F(t) = 0$
Spectral independence
$G = (V, E)$
u
$\sum_{(v,b)} | P(\tau(v) = b \mid \tau(u) = a) - P(\tau(v) = b) |$





Interesting/non-trivial even on trees.



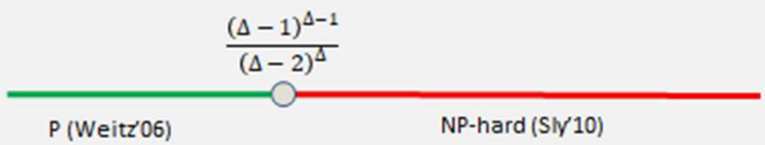
Hard-core model:

Undirected graph $G = (V, E)$ of maximum degree $\le \Delta$, parameter: $\lambda > 0$

distribution on independent sets of G

$$P(S) \propto \lambda^{|S|}$$

Can we efficiently sample from the distribution?



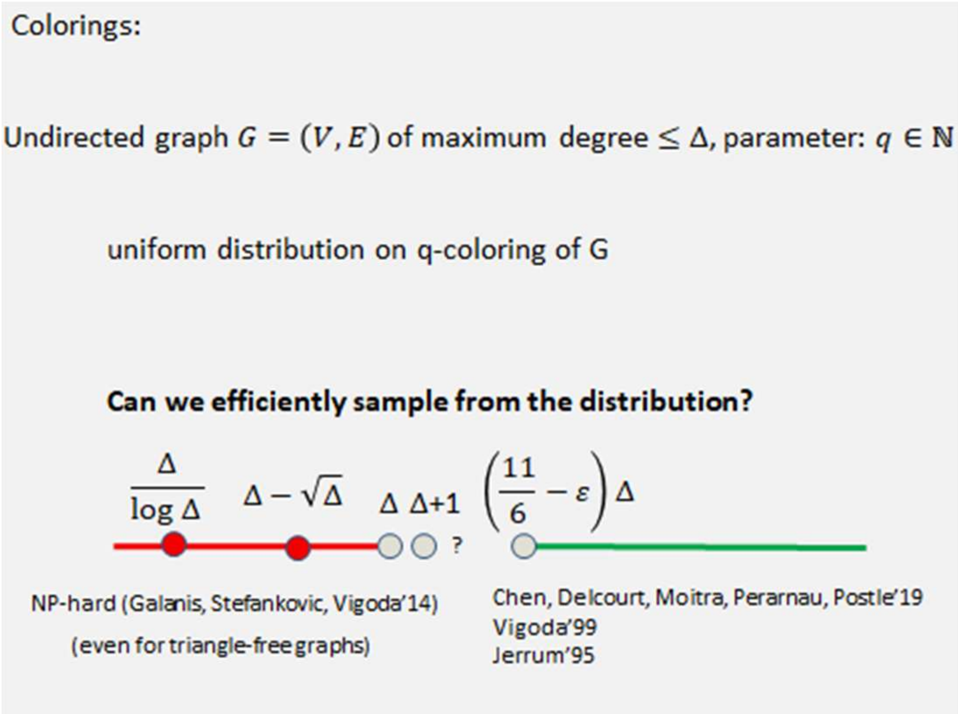
We understand everything (except the critical point):

WSM on trees
SSM on trees

SSM on trees ⇒ SSM on general graphs
spectral independence

All for

$$\lambda < \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$$



We understand something

WSM on trees for $q \geq \Delta + 1$ (Jonasson 2002)

SSM on trees for $q \geq 1.6\Delta$ (Efthymiou, Galanis, Hayes, Stefankovic, Vigoda 2019)

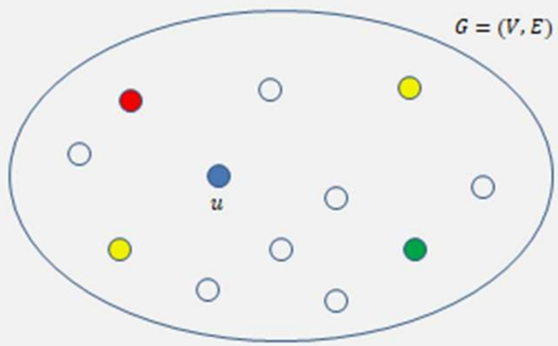
SSM, spectral independence on general graphs for $q \geq 2\Delta$

SSM, spectral independence on triangle-free graphs for $q \geq 1.74\Delta$

(Gamarnik, Katz, Misra 2015, Chen, Galanis, Stefankovic, Vigoda 2020)

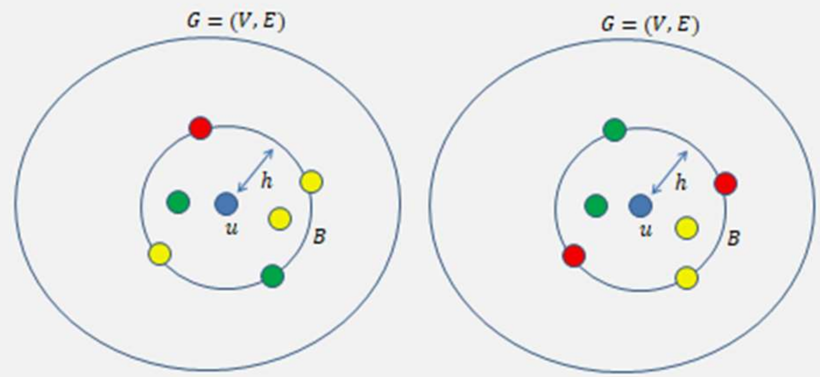
Formal connection?

Spectral independence



$$\sum_{(v,b)} |P(\tau(v) = b \mid \tau(u) = a) - P(\tau(v) = b)|$$

Strong spatial mixing



$$\max| P(\sigma(u) = a) \,|\sigma(B) = b_1) - P(\sigma(u) = a|\sigma(B) = b_2) \,| \le F(dist(u, D))$$

$$D = \{x \in S | b_1(x) \neq b_2(x)\}$$

$$\lim_{h \to \infty} F(t) = 0$$

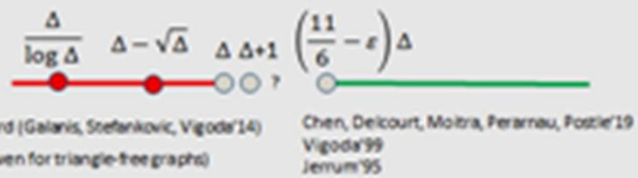
Improvements for trees? For general graphs?

Colorings:

Undirected graph $G = (V, E)$ of maximum degree $\leq \Delta$, parameter: $q \in \mathbb{N}$

uniform distribution on q-coloring of G

Can we efficiently sample from the distribution?



We understand something

WSM on trees for $q \geq \Delta + 1$ (Jonasson 2002)

SSM on trees for $q \geq 1.6\Delta$ (Efthymiou, Galanis, Hayes, Stefankovic, Vigoda 2019)

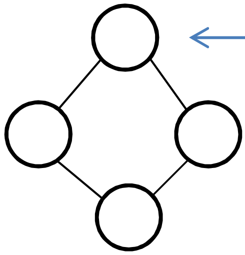
SSM, spectral independence on general graphs for $q \geq 2\Delta$

SSM, spectral independence on triangle-free graphs for $q \geq 1.74\Delta$

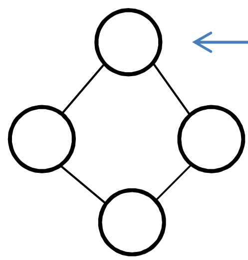
(Gamarnik, Katz, Misra 2015, Chen, Galanis, Stefankovic, Vigoda 2020) for hard-core model

SSM on trees $\Rightarrow$ SSM on general graphs

(Weitz'06)

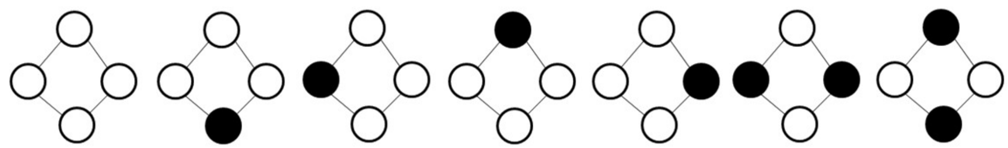


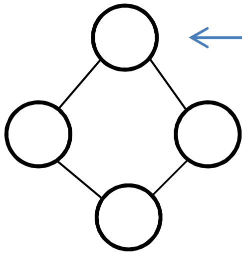
probability that u is occupied ?



probability that u is occupied ?

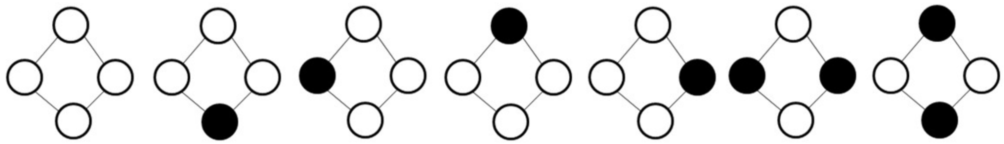
= 2/7

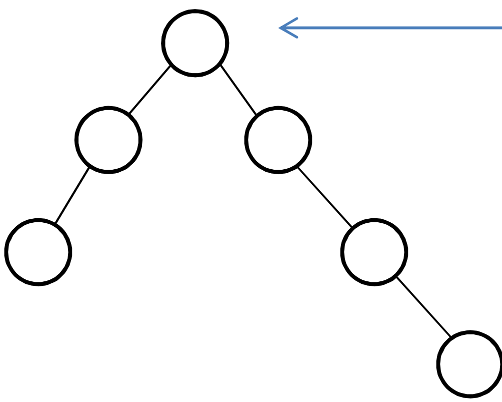




probability that u is occupied ?

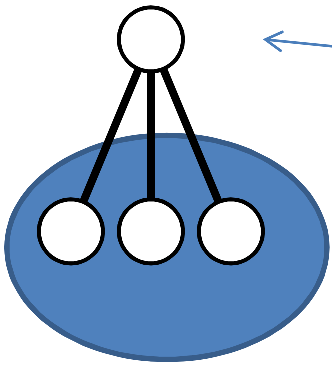
$$= 2/7$$



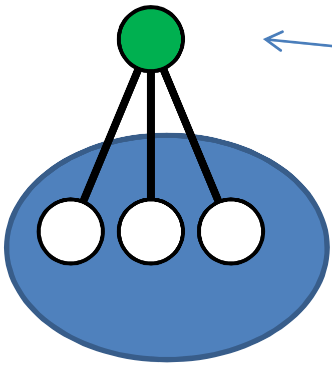


probability that u is occupied ?

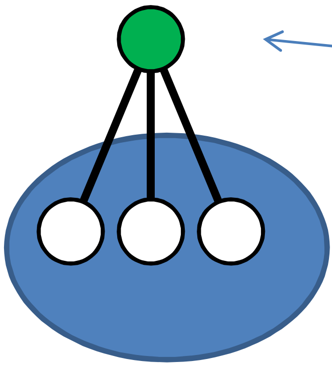
$$\frac{F_3 * F_4}{F_8} = \frac{2 * 3}{21} = 2/7$$



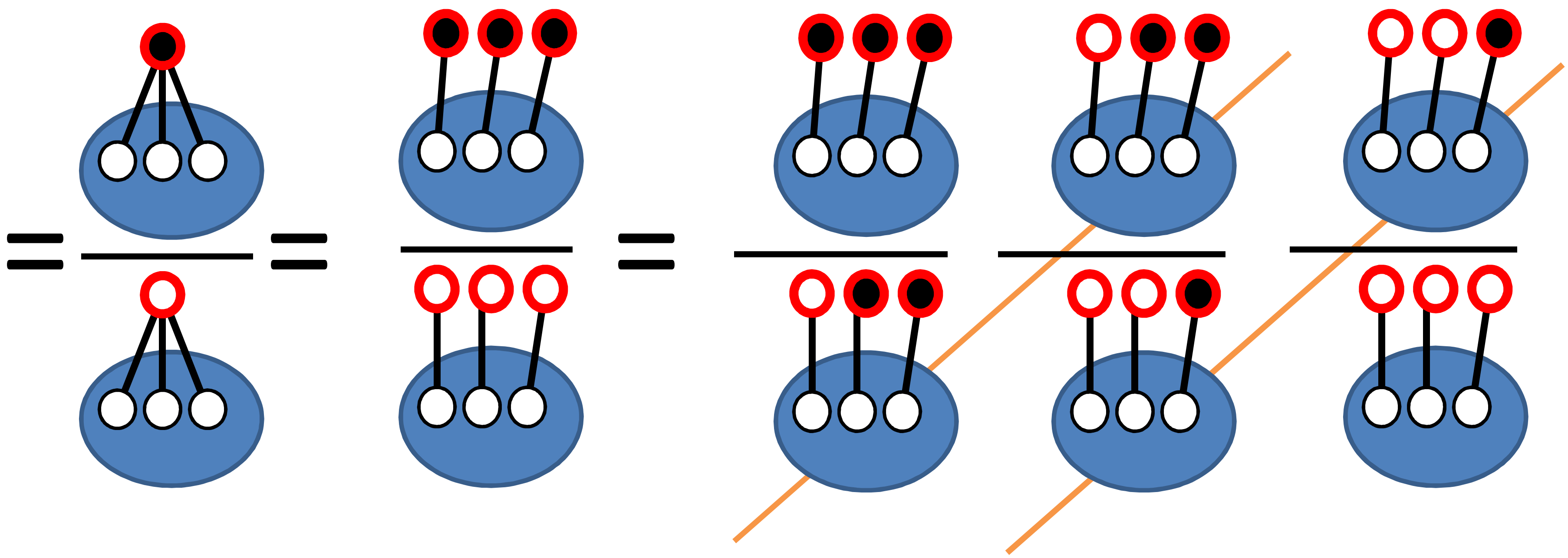
probability that u is occupied ?

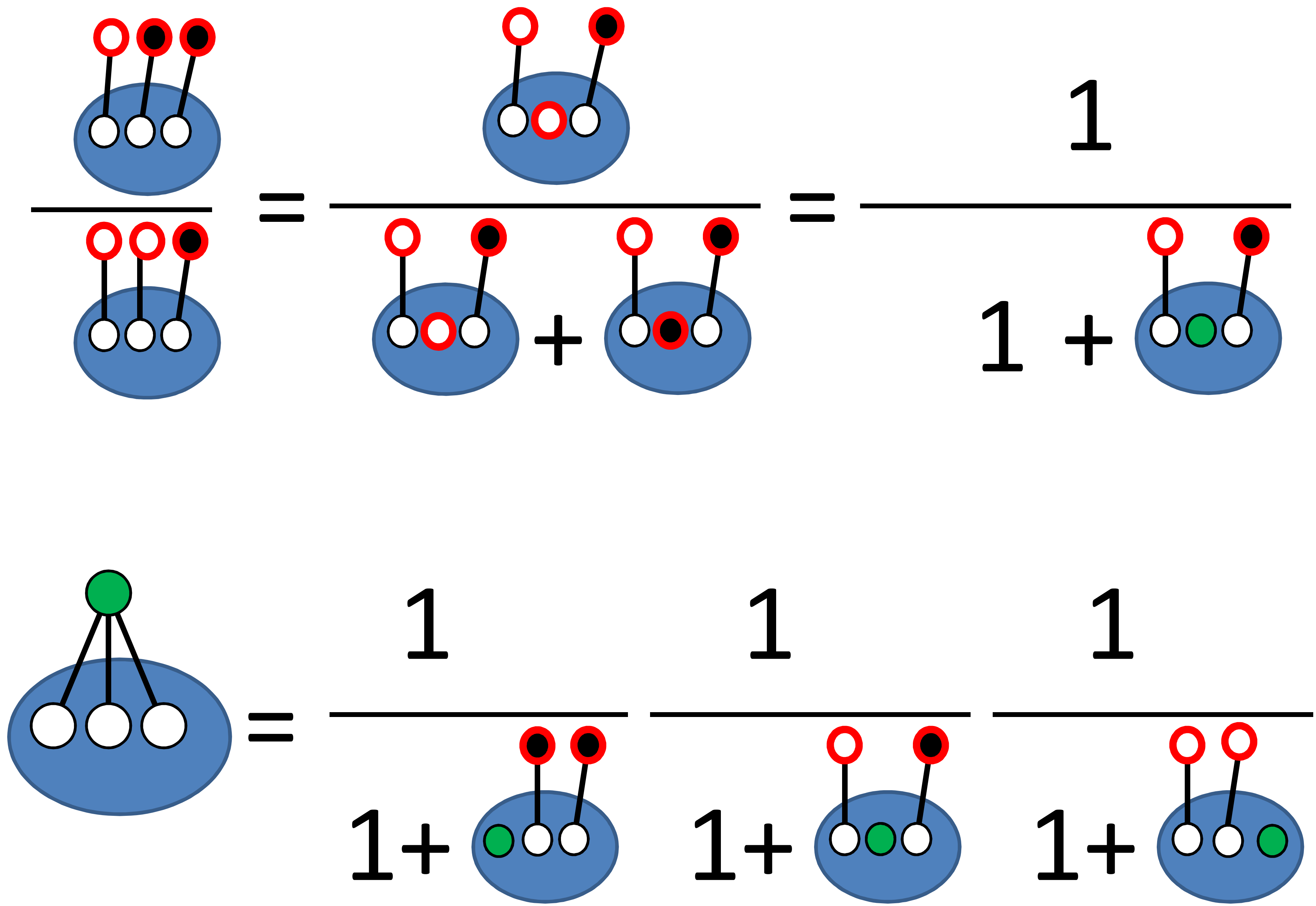


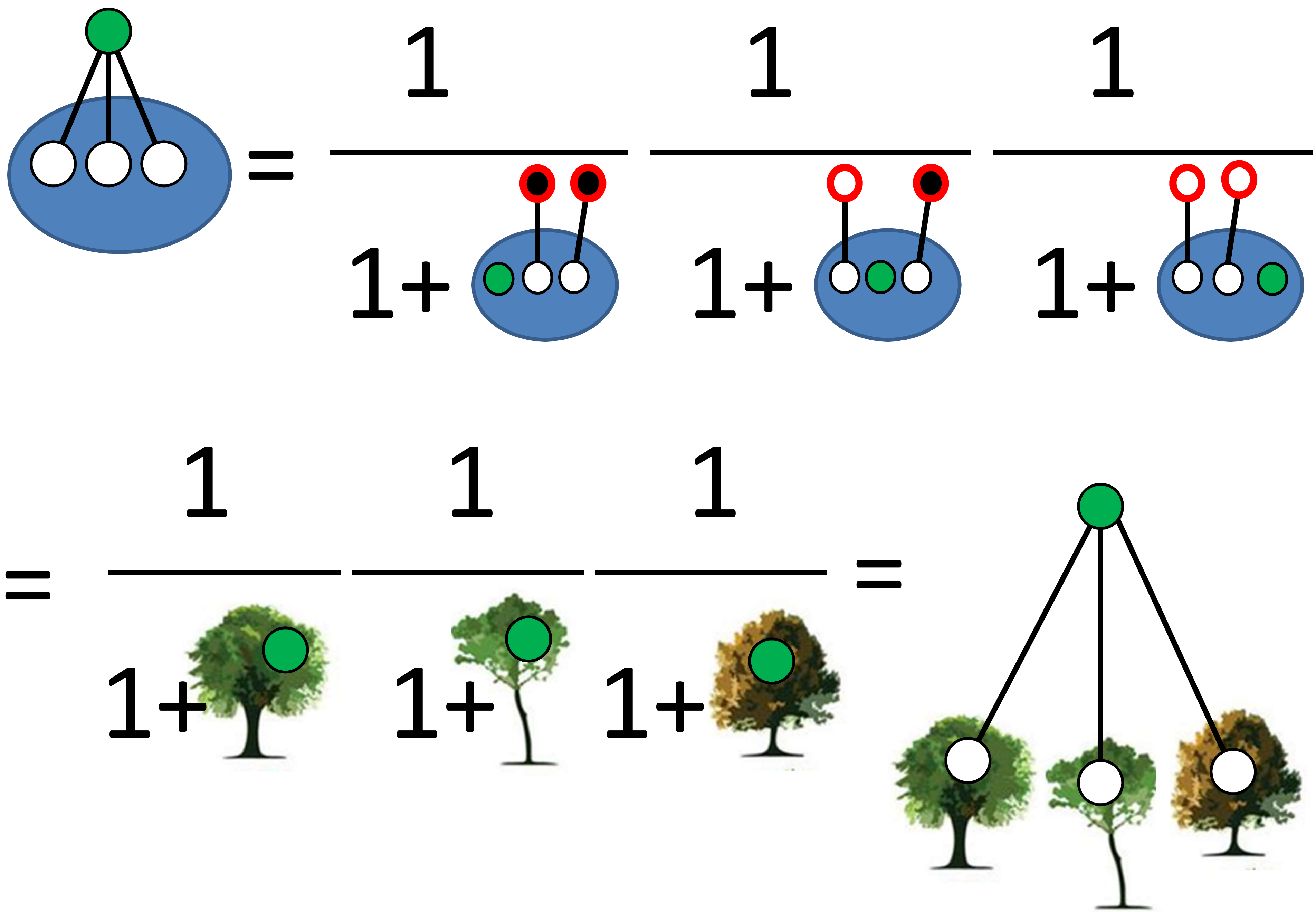
$$\frac{\text{probability that u is occupied}}{\text{probability that u is not occupied}} = r = \frac{p}{1-p}$$



$$\frac{\text{probability that u is occupied}}{\text{probability that u is not occupied}}$$







=
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for hard-core model (+general anti-ferro 2-spin models)

Spectral independence on trees $\Rightarrow$ spectral independence on general graphs

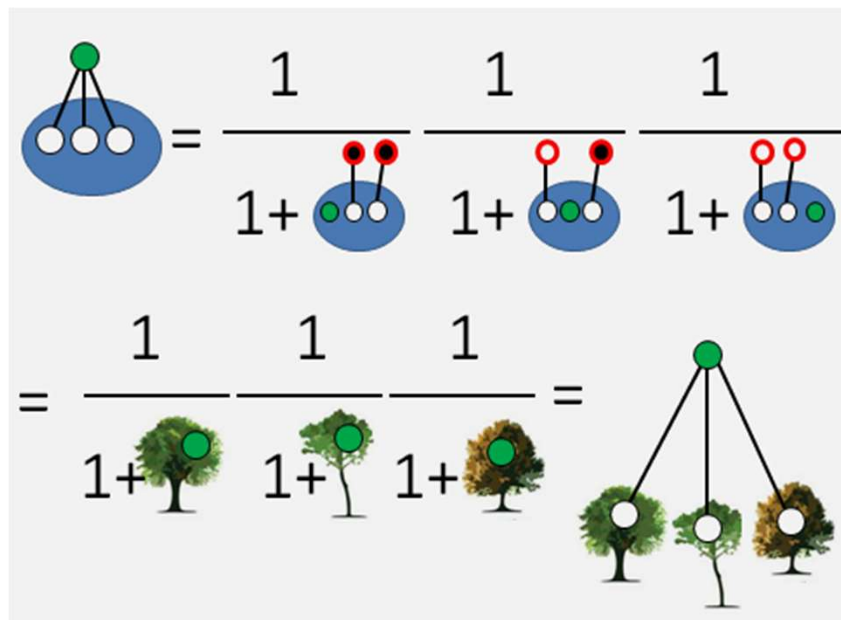
(Chen, Liu, Vigoda 2020)

Influence of u on v

$$\mathrm{Inf}_G^W(u \to v) = \sum_{(v,b)} |\, P(\tau(v) = 1 \mid \tau(u) = 1, W) - P(\tau(v) = 1 \mid \tau(u) = 0, W)|$$

$$R_G^W(u) = \frac{P(u = 1|W)}{P(u = 0|W)}$$

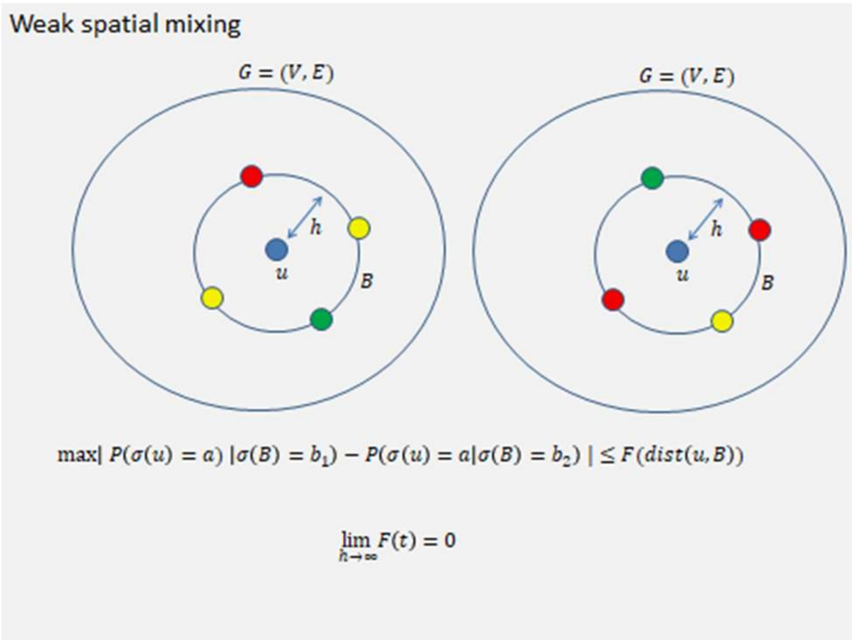
$$\lambda_v \frac{\partial}{\partial \lambda_v} \log \mathrm{R}_{\mathrm{G}}^{\mathrm{W}}(\mathrm{u}) = \mathrm{Inf}_G^W(u \to v)$$

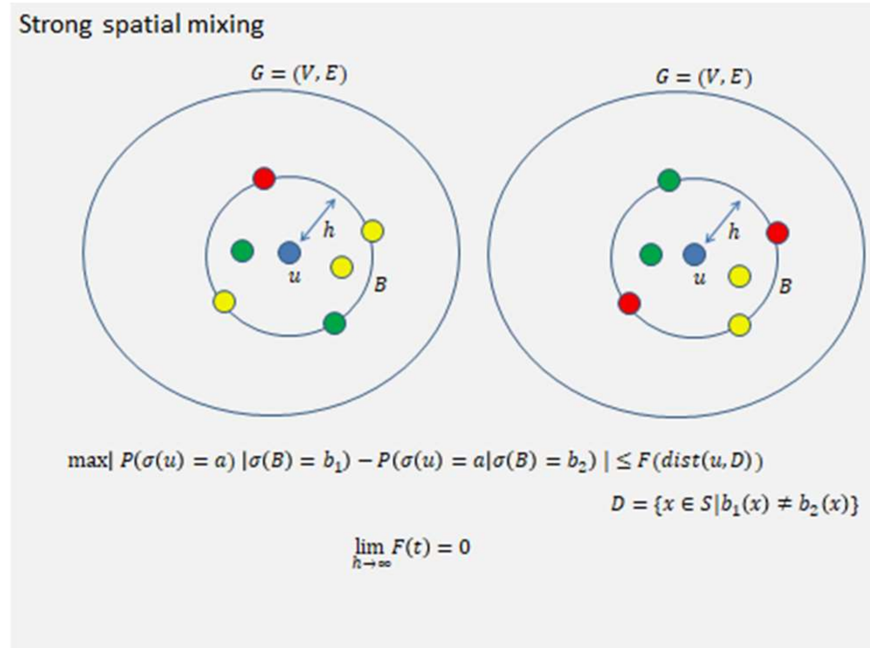


for hard-core model (+general anti-ferro 2-spin models)

establishing strong spatial mixing on trees

(Weitz 2006, Li, Lu, Yin 2013, Sinclair, Srivastava, Thurley 2014)





$$x = F(x_1, \ldots, x_d)$$

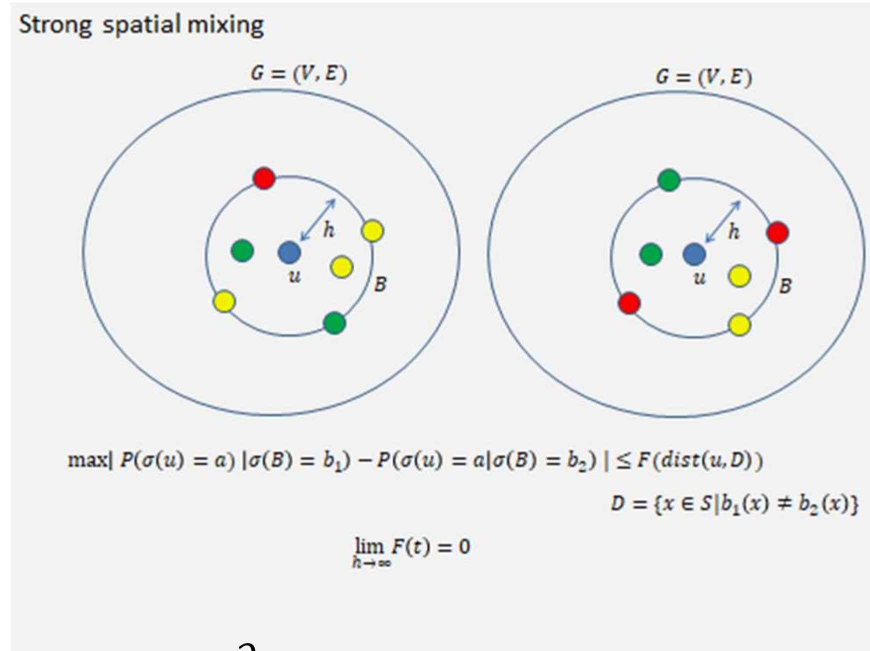
converges to a fixpoint

$$x = F(x_1, \ldots, x_d)$$

$$y = F(y_1, \ldots, y_d)$$

get closer together

$$F(x_1, \ldots, x_n) = \frac{\lambda}{(1 + x_1) \ldots (1 + x_d)}$$



$$(x_1, \Delta x_1), \ldots, (x_d, \Delta x_d) \rightarrow F(x_1, \ldots, x_n), \Delta x_1 \frac{\partial}{\partial x_1} F + \cdots + \Delta x_d \frac{\partial}{\partial x_d} F$$

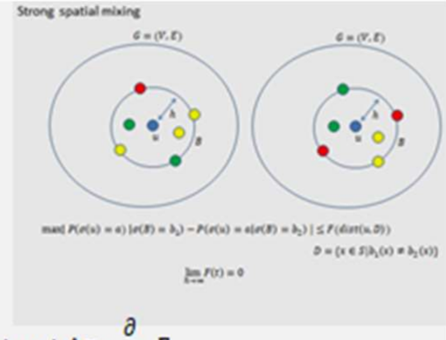
the perturbation gets shorter

$$x = F(x_1, \ldots, x_d)$$

$$y = F(y_1, \ldots, y_d)$$

get closer together

$$F(x_1, \dots, x_n) = \frac{\lambda}{(1 + x_1) \dots (1 + x_d)}$$



$$(x_1, \Delta x_1), \dots, (x_d, \Delta x_d) \to F(x_1, \dots, x_n), \Delta x_1 \frac{\partial}{\partial x_1} F + \cdots + \Delta x_d \frac{\partial}{\partial x_d} F$$

the perturbation gets shorter

$$x = F(x_1, \dots, x_d)$$

$$y = F(y_1, \dots, y_d)$$

get closer together

$$\Phi'(x) = \big(x(1 + x)\big)^{1/2}$$

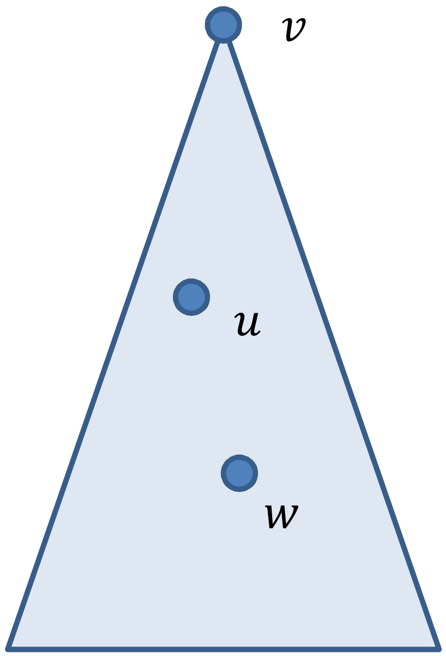
$$\big|\big|\nabla(\Phi \circ F \circ \Phi^{-1})(x_1, \dots, x_d)\big|\big|_1 \le \tau < 1$$

for hard-core model (+general anti-ferro 2-spin models)

establishing spectral independence on trees

(Chen, Liu, Vigoda 2020)

$$\mathrm{Inf}_G^W(u \to v) = \sum_{(v,b)} |P(\tau(v) = 1 \mid \tau(u) = 1, W) - P(\tau(v) = 1 \mid \tau(u) = 0, W)|$$



$$\mathrm{Inf}_G^W(v \to w) = \mathrm{Inf}_G^W(v \to u)\ \mathrm{Inf}_G^W(u \to w)$$

$$h(\log \mathrm{R}_\mathrm{G}^\mathrm{W}(\mathrm{v})) = \mathrm{Inf}_G^W(u \to v)$$

$$\sum_v \mathrm{Inf}_G^W(r \to v) = \max\left|\frac{h(\log(R(r_i)))}{\psi(\log(R(r_i)))}\right| .\max \psi(\log(R(r_i))) \sum_v \mathrm{Inf}_G^W(r_i \to v)$$

Decays by factor τ

$$||\nabla(\Phi \circ F \circ \Phi^{-1})(x_1, \dots, x_d)||_1 \leq \tau < 1$$

Colorings

(Chen, Galanis, Stefankovic, Vigoda 2020)

Colorings

connection with trees missing

“computational tree recursion” (using list colorings)

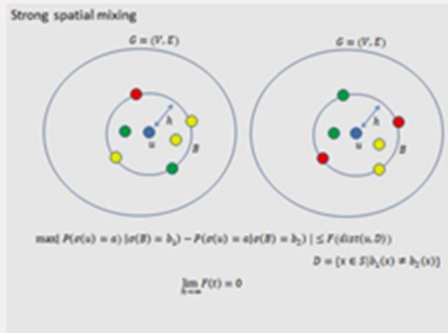
$$\mathcal{I}[v \to (w,k)] = \max_{i,j\in[q]} |\mathbb{P}(\sigma_w = k \mid \sigma_v = i) - \mathbb{P}(\sigma_w = k \mid \sigma_v = j)|$$

$$\mathcal{I}^*_{G,\mathcal{L}}(v) = \frac{1}{\Delta_G(v)} \sum_{w\in V\setminus\{v\}} \sum_{k\in[q]} \mathcal{I}_{G,\mathcal{L}}[v \to (w,k)]$$

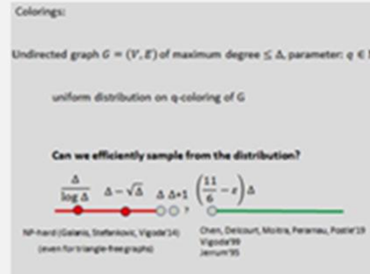
$$\mathcal{I}^*_{G,\mathcal{L}}(v) \le \max_{u\in N_G(v)} \left\{ R_{G_v,\mathcal{L}_v}(u)\big(\Delta_{G_v}(u) \cdot \mathcal{I}^*_{G_v,\mathcal{L}_v}(u) + q\big)\right\},$$

where $R_{G_v,\mathcal{L}_v}(u) = \max_{L\in\mathcal{L}_v} \max_{c\in L(u)} \frac{\mathbb{P}_{G_v,L}(\sigma_u = c)}{\mathbb{P}_{G_v,L}(\sigma_u \neq c)}$ *for* $u \in N_G(v)$.

Formal connection?



Improvements for trees? For general graphs?



We understand something

WSM on trees for $q \geq \Delta + 1$ (Jonasson 2002)
SSM on trees for $q \geq 1.6\Delta$ (Efthymiou, Galanis, Hayes, Stefankovic, Vigoda 2019)
SSM, spectral independence on general graphs for $q \geq 2\Delta$
SSM, spectral independence on triangle-free graphs for $q \geq 1.74\Delta$
(Gamarnik, Katz, Misra 2015, Chen, Galanis, Stefankovic, Vigoda 2020)

Entropic independence $\mathcal{D}_{\mathrm{KL}}(\nu D_{k\to 1} \parallel \mu D_{k\to 1}) \leq \frac{1}{\alpha k} \mathcal{D}_{\mathrm{KL}}(\nu \parallel \mu).$
Anari, Jain, Koehler, Pham, Vuong 2020

Fractional log-concavity, sector stability
Alimohammadi, Anari, Shiragur, Vuong 2020

$\log f(x_1^\alpha, \ldots, x_n^\alpha)$ is log concave

Entropy factorization, log Sobolev constants
Chen, Liu, Vigoda 2020