spectral independence, sampling bases of matroids, HDX

Daniel Štefankovič

Kauffman, Oppenheim’18 (High order random walks: beyond spectral gap)

Oppenheim’18 (Local spectral expansion approach to HDX)

Anari, Liu, Oveis Gharan, Vinzant’19 (Log-Concave polynomials I-II-III-IV)

Alev, Lau’20 (Improved analysis of higher order random walks and applications)

Lau’22 (https://cs.uwaterloo.ca/~lapchi/cs860/notes.html)

Anari, Liu, Oveis Gharan’20 (Spectral independence and HDX and applications to the hardcore model)

Chen, Liu, Vigoda’20 (Rapid Mixing of Glauber Dynamics up to Uniqueness via Contraction)

Chen, Galanis, Stefankovic, Vigoda’20 (Rapid Mixing for Colorings via Spectral Independence)

random walks on hypergraphs (simplicial complexes)

background

inductive approach to bound spectral gap

Distribution π_r on (all) size-r subsets of $[n]$

$n = 3$ $r = 2$

$\{1, 2\}$ $\{2, 3\}$ $\{1, 3\}$

$1/2$ $1/2$ $\emptyset$

Can we sample from π_r efficiently?

negative example: min-bisection

$$G = (V, E), |V| = n, r = \frac{n}{2}$$

$\pi_r(A) \propto 1$ if $|E(A, V \setminus A)| \leq T$

$\pi(A) = 0$ otherwise

Distribution π_r on (all) size-r subsets of $[n]$

Can we sample from π_r efficiently?

example: spanning trees

$$G = (V, E), |E| = n, r = n - 1$$

$\pi(A) \propto 1$ if (V, A) is a tree

$\pi(A) = 0$ otherwise

Distribution π_r on (all) size-r subsets of $[n]$

Can we sample from π_r efficiently?

Distribution π_r on all size-r subsets of $[n]$

STEP 1:

remove a uniformly random element $a \in X_t$, let $\mathrm{T} = X_t \setminus \{a\}$

STEP 2:

$$P(b) \propto \pi_r(T \cup \{b\})$$

add a random element b
(so that the stationary distribution is π_r)

TODO: detailed balance, reversibility

Distribution π_r on all size-r subsets of $[n]$

STEP 1:

remove a uniformly random element $a \in X_t$, let $T = X_t \setminus \{a\}$

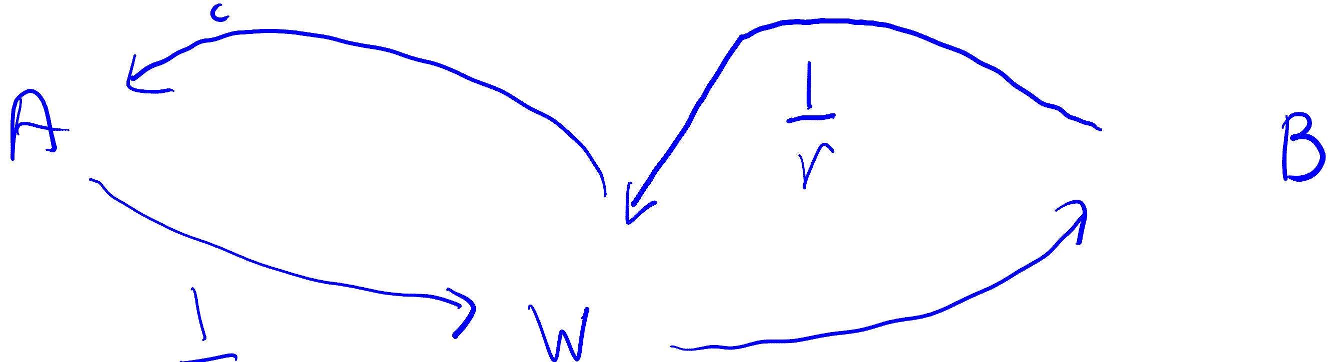
STEP 2:

add a random element b $\quad P(b) \propto \pi_r(T \cup \{b\})$
(so that the stationary distribution is π_r.)

$$\pi(A)\, P(A, B) = \pi(B)\, P(B, A)$$

Let's check the detailed balance condition

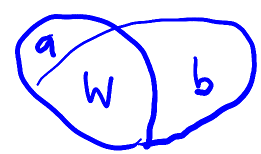
$$\frac{\pi_r(A)}{\sum_c \pi_r(W \cup \{c\})}$$



$$\frac{1}{r}$$

$$\frac{1}{r}$$

$$\frac{\pi_r(B)}{\sum_c \pi_r(W \cup \{c\})}$$



Example: weighted spanning trees of (G, w)

$$P(T) \propto \prod_{e \in T} w(e)$$

Distribution π_r on all size-r subsets of $[n]$

STEP 1:

remove a uniformly random element $a \in X_t$, let $\mathrm{T} = X_t \setminus \{a\}$

STEP 2:

add a random element b $\quad P(b) \propto \pi_r(T \cup \{b\})$

(so that the stationary distribution is π_r.)

Distribution π_r on all size-r subsets of $[n]$

STEP 1:

remove a uniformly random element $a \in X_t$, let $\mathrm{T} = X_t \setminus \{a\}$

STEP 2:

$$P(b) \propto \pi_r(T \cup \{b\})$$

add a random element b
(so that the stationary distribution is π_r)

Does the above MC mix rapidly?

(Kauffman, Oppenheim’18, Anari, Liu, Oveis Gharan, Vinzant’19,
Alev, Lau’20, Anari, Liu, Oveis Gharan’20)

Distribution π_r on all size-r subsets of $[n]$

Distribution π_{r-1} on all size-$(r-1)$ subsets of $[n]$

$$\pi_{r-1}(A) = \frac{1}{r} \sum_{B;\, A \subseteq B} \pi_r(B)$$

Distribution π_r on all size-r subsets of $[n]$

STEP 1:

remove a uniformly random element $a \in X_t$, let $\mathrm{T} = X_t \setminus \{a\}$

STEP 2:

$P(b) \propto \pi_r(T \cup \{b\})$

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Does the above MC mix rapidly?

(Kauffman, Oppenheim'18, Anari, Liu, Oveis Gharan, Vinzant'19, Alev, Lau'20, Anari, Liu, Oveis Gharan'20) random walks on hypergraphs (simplicial complexes)

background

inductive approach to bound spectral gap

Reversible Markov chain with transition matrix P, stationary distribution π

DP is symmetric $\qquad D = \text{diag}(\pi)$

$$\pi(A) P(A,B) = \pi(B) P(B,A)$$

$$(DP)_{A,B} = \pi(A) P(A,B)$$

$$\pi^T P = \pi^T$$

$$P\mathbb{1} = 1$$

Reversible Markov chain with transition matrix P, stationary distribution π

for symmetric matrix A $\lambda_1(A) = \max \frac{x^T A x}{x^T x}$ (Rayleigh quotient)

$A = SQS^{-1}$

S is orthogonal

Q is diagonal with eigenvalues on diagonal

$\max \frac{x^T S Q S^{-1} x}{x^T S S^{-1} x} =$

$S^{-1} = S^T$

$= \max \frac{y^T Q y}{\|y\|_2^2}$

$\|Sx\|_2 = \|x\|_2$

Reversible Markov chain with transition matrix P,stationary distribution π

mixing time controlled by $\lambda_1(P - 1^T\pi) = \lambda_2(P)$

$$\frac{P+I}{2}$$

$$\tau_{mix} \leq \frac{1}{1-\lambda_2} \left| \log \pi_{min} \right|$$

$D^{1/2}(P - 1^T\pi)D^{-1/2} \sim (P - 1^T\pi)$

TODO: similar ⇒ spectrum

$$\det(Ix - A)$$

$$\det(B(Ix - A)B^{-1}) = \det(B)\det(Ix - A)\det(B^{-1}) = \det(Ix - A)$$

$$(DP - \pi\pi^T) = D^{1/2}D^{1/2}(P - 1^T\pi)D^{-1/2}D^{1/2}$$

Reversible Markov chain with transition matrix P, stationary distribution π

DP is symmetric $\quad D = \mathrm{diag}(\pi)$

mixing time controlled by $\quad \lambda_1(P - 1^T\pi) = \lambda_2(P)$

for symmetric matrix A $\quad \lambda_1(A) = \max \frac{x^T A x}{x^T X}$ $\quad$ (Rayleigh quotient)

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2})D^{1/2}(P - 1^T\pi)D^{-1/2}(D^{1/2}f)$$

$$\sum_{a,b} f(a)f(b)\ (\pi(a)P(a,b) - \pi(a)\pi(b))$$

$$f^T D f = (f^T D^{1/2})(D^{1/2}f)$$

$$\sum_{a} f(a)^2 \pi(a)$$

$\lambda_2 < 1 \Leftrightarrow$ state space connected

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2}) D^{1/2}(P - 1^T\pi) D^{-1/2}(D^{1/2} f)$$

$$\sum_{a,b} f(a)f(b)\ (\pi(a)P(a,b) - \pi(a)\pi(b))$$

$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_a f(a)^2 \pi(a)$$

$$\lambda_2(P) = \max \frac{x^T(DP - \pi\pi^T)x}{x^T DX}$$ (Rayleigh quotient)

Eigenvector for eigenvalue 1 of DP, D: 1
Eigenvector for eigenvalue 0 of $DP - \pi\pi^T, D$: 1

Eigenvectors for different eigenvalues are perpendicular: $\langle 1, x\rangle_\pi = \pi^T x = 0$

$$DP\ x = \lambda_2 D\ x$$

$$\lambda_2(P) = \max \frac{x^T(DP - \pi\pi^T)x}{x^T DX}$$ (Rayleigh quotient**)**

Eigenvector for eigenvalue 1 of DP, D: 1
Eigenvector for eigenvalue 0 of $DP - \pi\pi^T, D$: 1

Eigenvectors for different eigenvalues are perpendicular: $\langle 1, x\rangle_\pi = \pi^T x = 0$

$$DP\,x = \lambda_2 D\,x$$

$$x^T(DP - \pi\pi^T)x \le \lambda_2 x^T D x$$

$$x^T(DP - \pi\pi^T)x \le \lambda_2 x^T(D - \pi\pi^T)x$$

$$x^T(DP - \pi\pi^T)x \leq \lambda_2 x^T D x$$

$x = 1$

$0 \leq \lambda_2$

$\pi(A)\pi(B)$

$(x^+)^T \cdot 1 = 0$

$$x^T(DP - \pi\pi^T)x \leq \lambda_2 x^T (D - \pi\pi^T) x$$

$0 \leq 0$ ← better

$x = 1 + x^+$

random walks on hypergraphs (simplicial complexes)

background

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Distribution π_r on all size-r subsets of $[n]$

STEP 1:

remove a uniformly random element $a \in X_t$, let $S = X_t \setminus \{a\}$

STEP 2:

add a random element b
(so that the stationary distribution is π_r) $P(b) \propto \pi_r(S \cup \{b\})$

Does the above MC mix rapidly?

(Kauffman, Oppenheim'18, Anari, Liu, Oveis Gharan, Vinzant'19,
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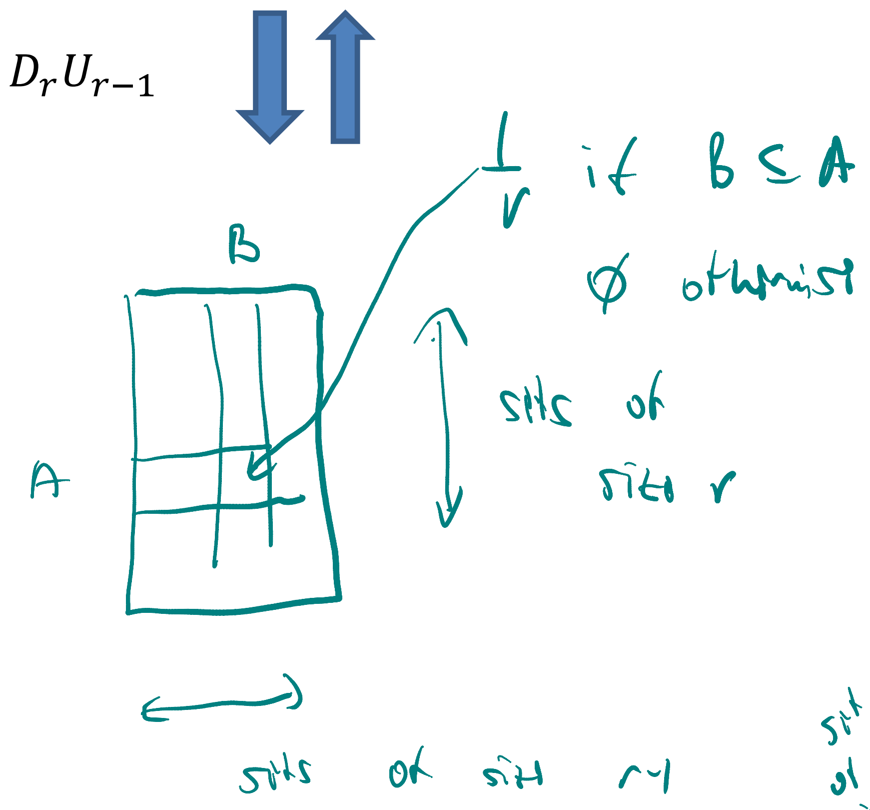
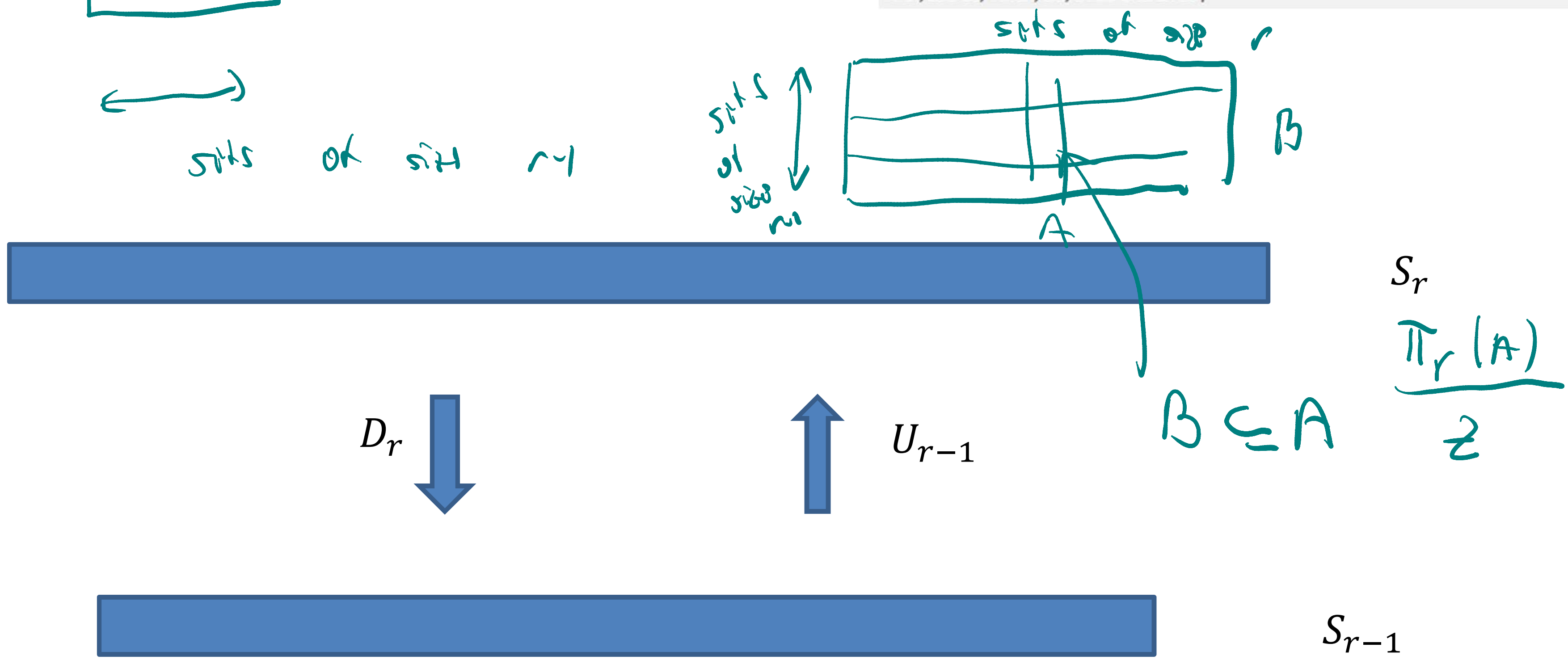
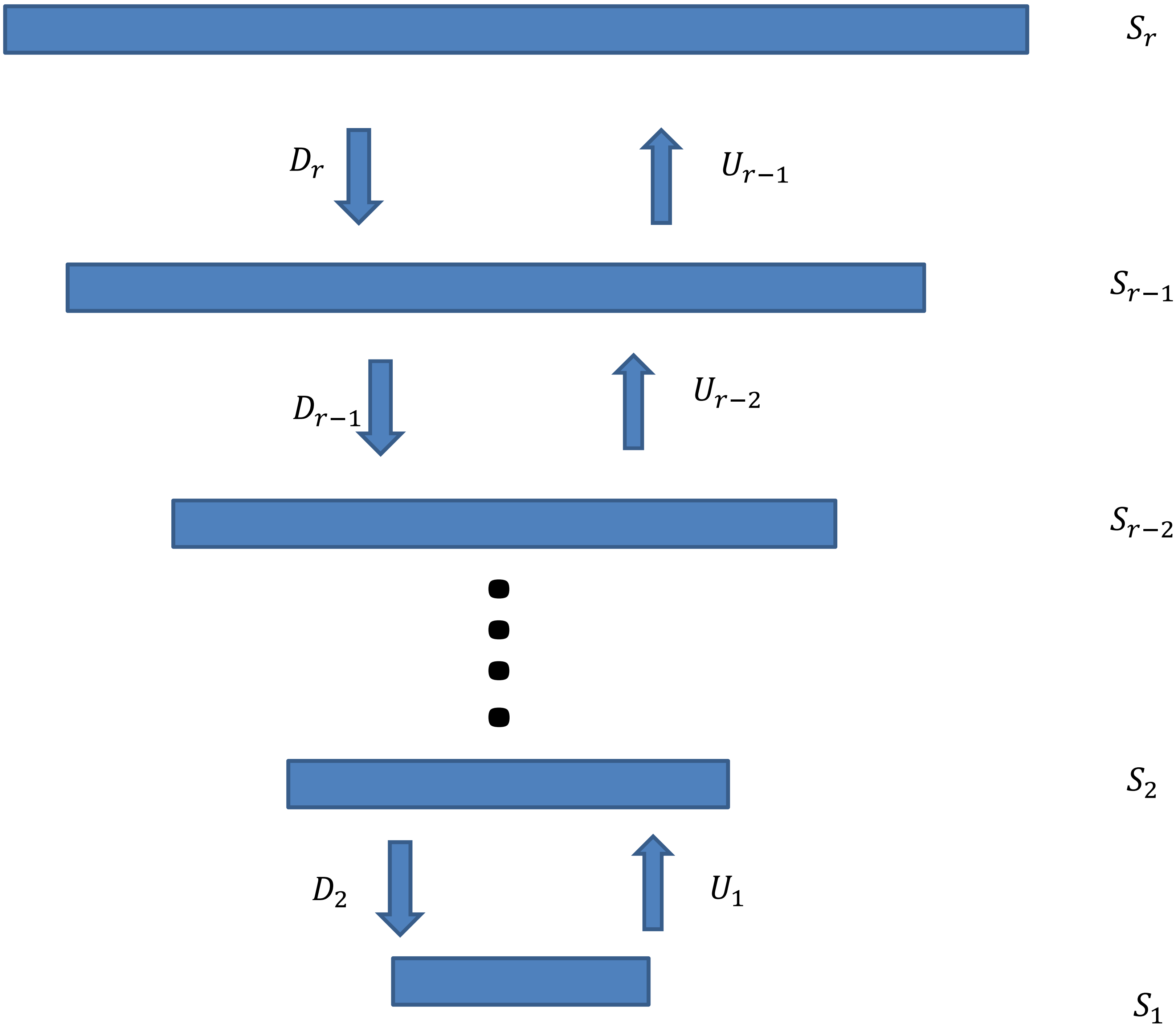
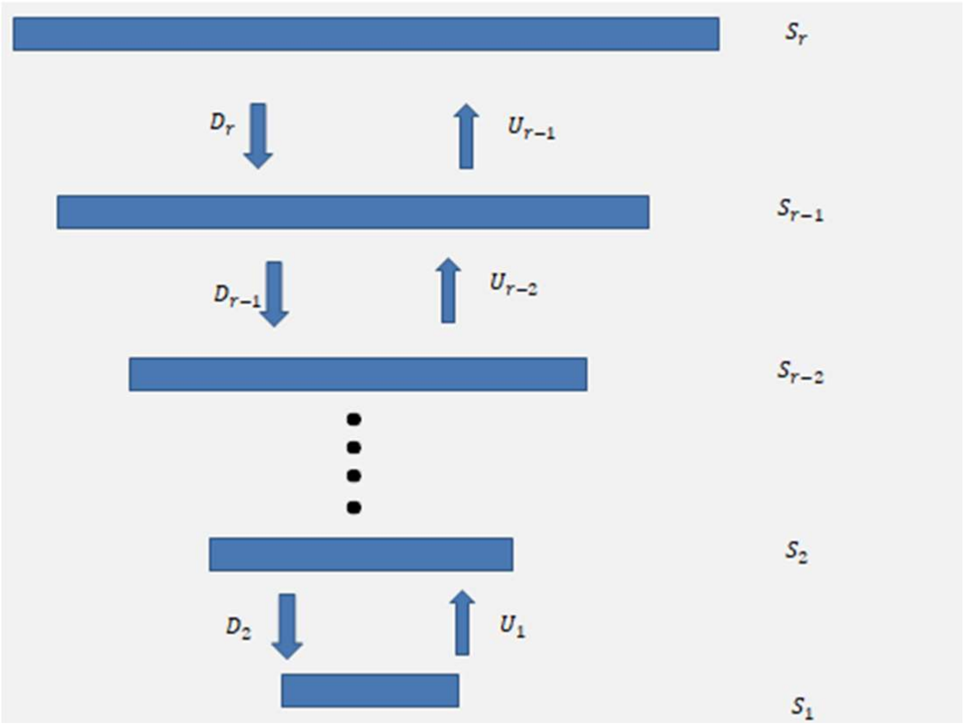
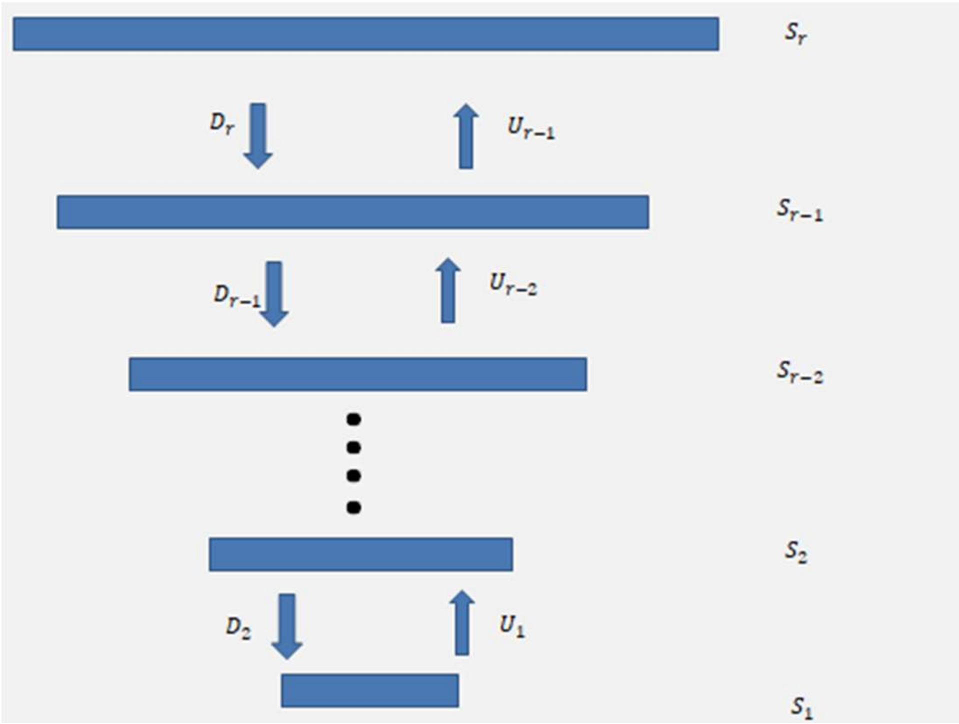





S_r
D_r
U_{r-1}
S_{r-1}
D_{r-1}
U_{r-2}
S_{r-2}
S_2
D_2
U_1
S_1

$$\pi_k(S) = \binom{r}{k}^{-1} \sum_{\substack{T;S\subseteq T \\ |T|=r}} \pi_r(T)$$





MAIN IDEA: analyze the spectrum inductively.

IDEA #1: $\mathrm{SPEC}_{\neq 0}(D_k U_{k-1}) = \mathrm{SPEC}_{\neq 0}(U_{k-1} D_k)$

IDEA #2: understand $(U_k D_{k+1})$ vs $(D_k U_{k-1})$

IDEA #1: $\mathrm{SPEC}_{\neq 0}(AB) = \mathrm{SPEC}_{\neq 0}(BA)$

A   m × n

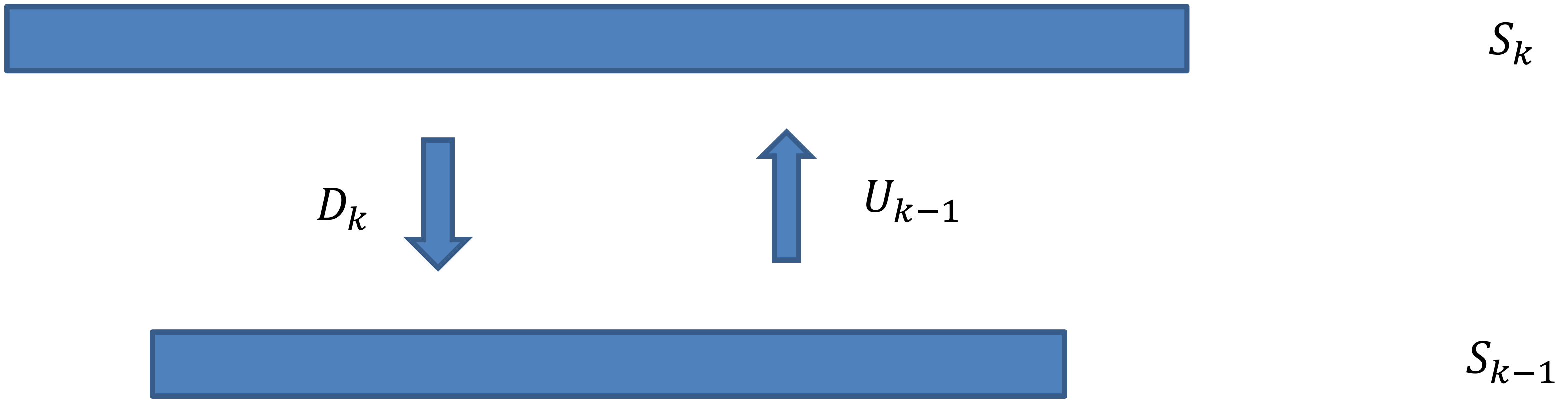
B   n × m

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

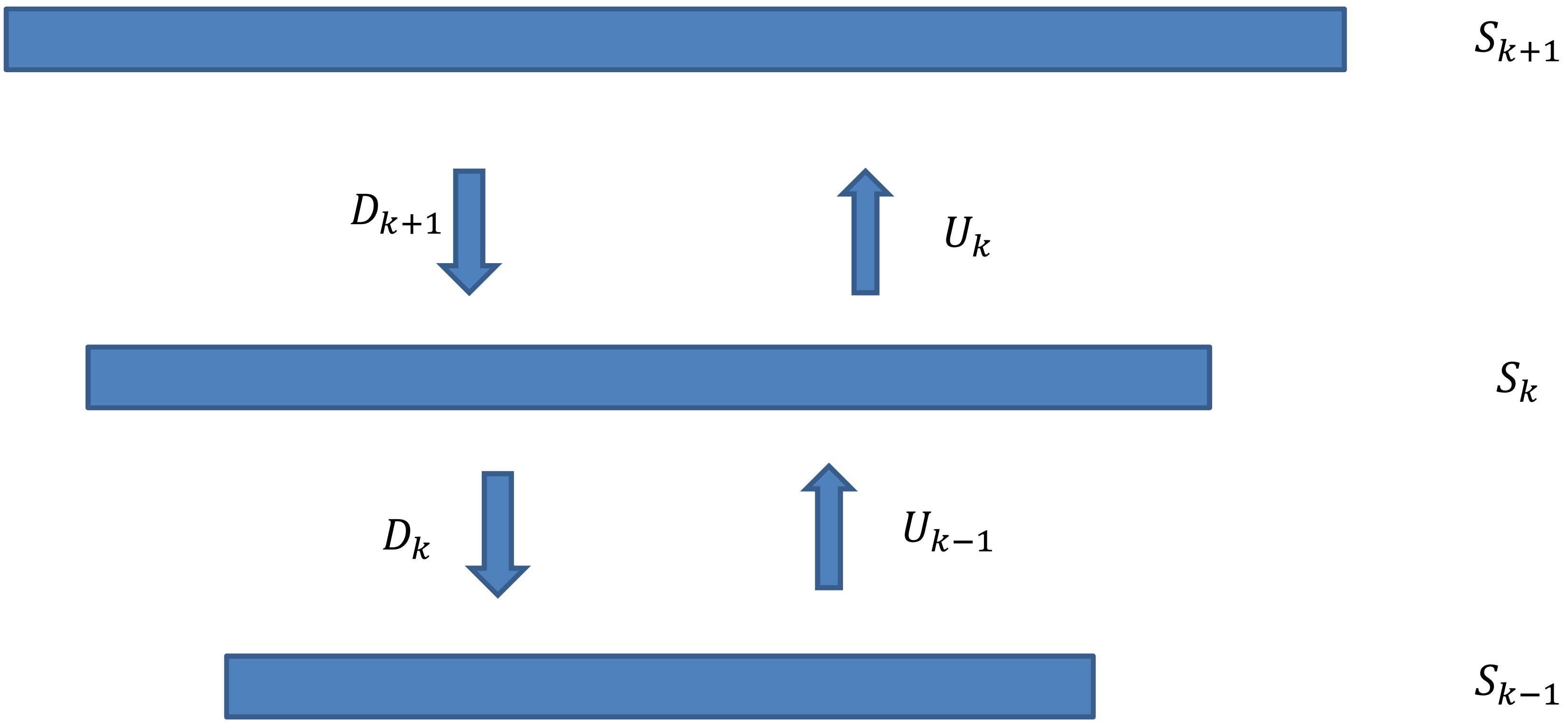
$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

$$\mathrm{char}(AB)x^n = \mathrm{char}(BA)x^m$$

IDEA #1: $\mathrm{SPEC}_{\neq 0}(D_k U_{k-1}) = \mathrm{SPEC}_{\neq 0}(U_{k-1} D_k)$

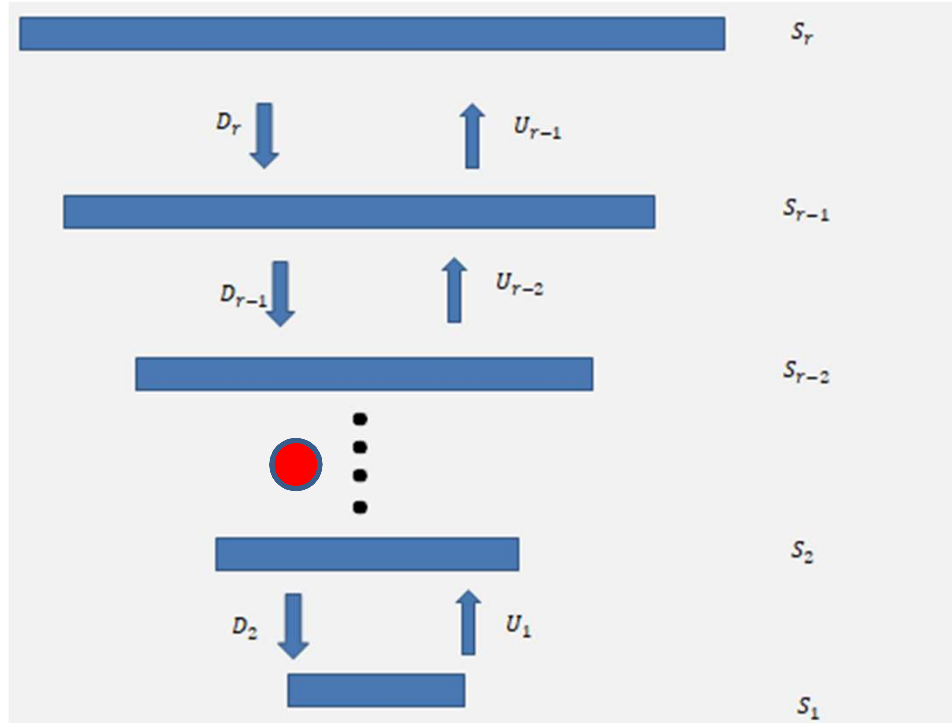


IDEA #2: understand $(U_k D_{k+1})$ vs $(D_k U_{k-1})$



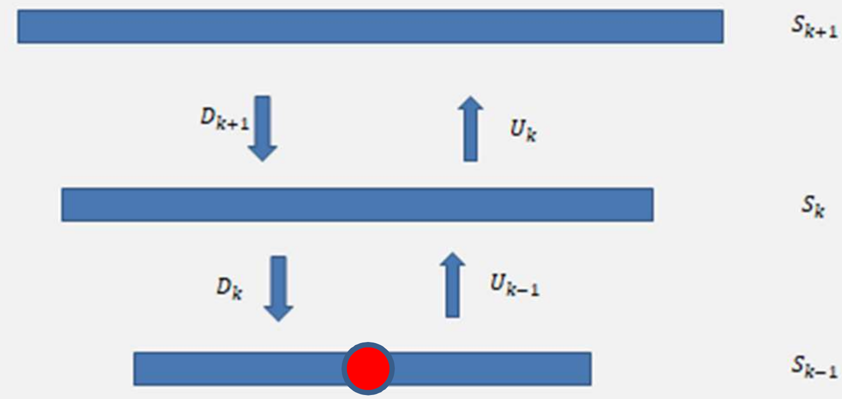
“Focusing on a certain subsets” (link)

$$\pi\ (A) = \frac{\pi(A \cup W)}{\binom{|A \cup W|}{|A|}\pi(W)}$$



$$\binom{r}{k}\ \pi_k(W) = \sum_{T;S \subseteq T} \pi_r(T)$$

IDEA #2: understand $(U_k D_{k+1})$ vs $(D_k U_{k-1})$



1) fix set W

2) consider the chain on sets $W \cup \{a\}$
with up-down transition (no self loops)

$$P(a,b) = \frac{\pi_{k+1}(W \cup \{a,b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2})D^{1/2}(P - 1^T\pi)D^{-1/2}(D^{1/2}f)$$

$$\sum_{a,b \notin W} f(W \cup \{a\}) f(W \cup \{b\}) \left(\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)} \right)$$

$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

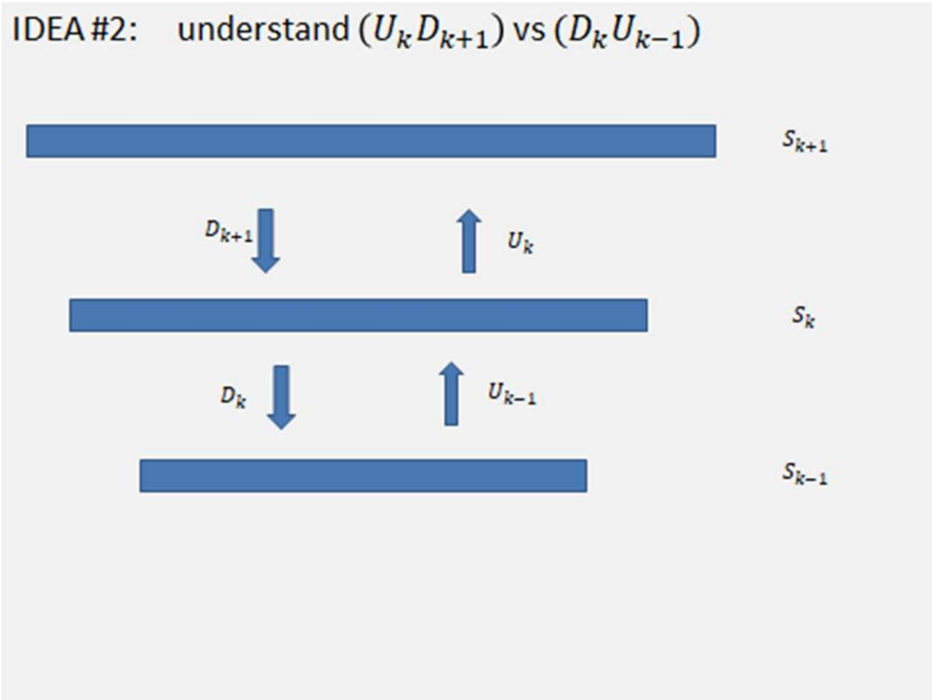
$$\sum_a f(W \cup \{a\})^2 \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

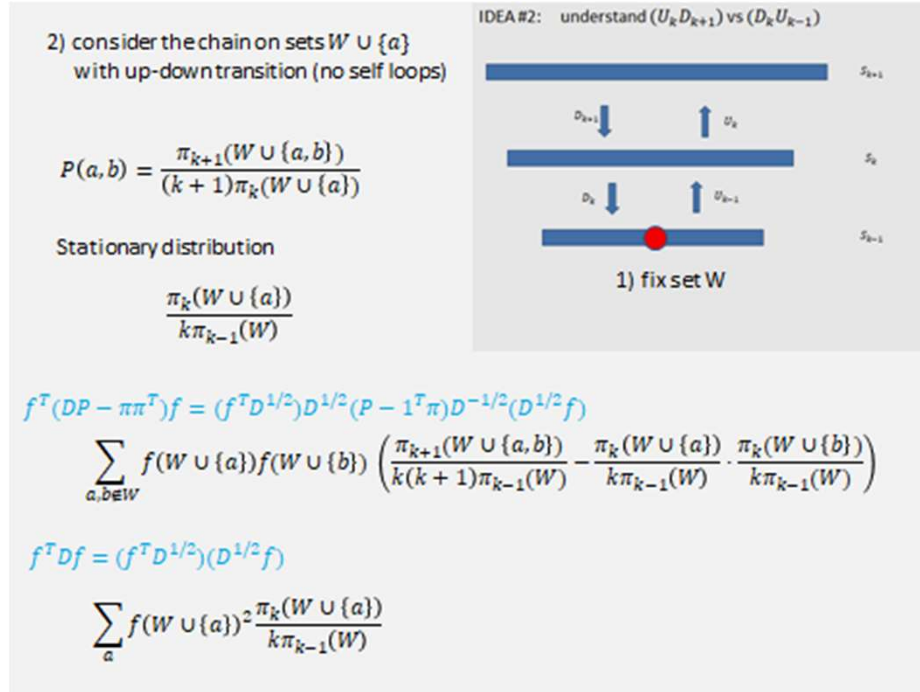
TODO: check detailed balance, check that distribution

$$P(a, b) = \frac{\pi_{k+1}(W \cup \{a, b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$





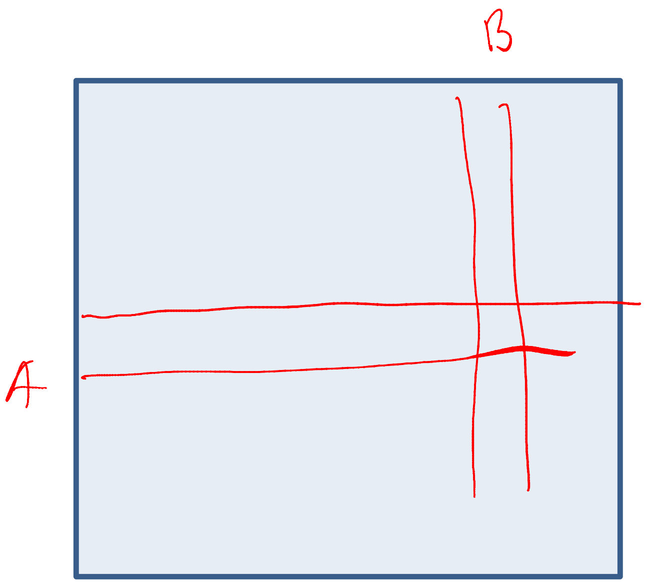
If the ``local'' chains all have $\lambda_2 \leq \gamma$

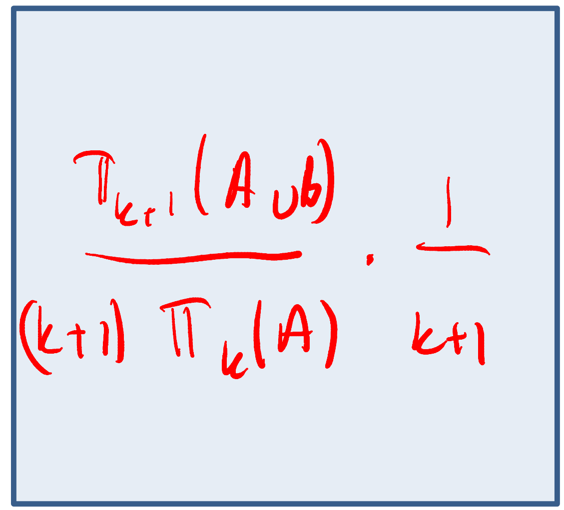
then $U_k D_{k+1}$ will not be much worse than $D_k U_{k-1}$

$D_k U_{k-1}$

$$\frac{1}{k} \cdot \frac{\pi_k(B)}{k\,\pi_{k-1}(A \cap B)}$$

$U_k D_{k+1}$





off-diagonal $|A \oplus B| = 2$

$$\frac{1}{k^2} \cdot \frac{\pi_k(B)}{\pi_{k-1}(A \cap B)}$$

diagonal

$$\frac{1}{k^2} \cdot \sum_{a \in A} \frac{\pi_k(A)}{\pi_{k-1}(A \setminus \{a\})}$$

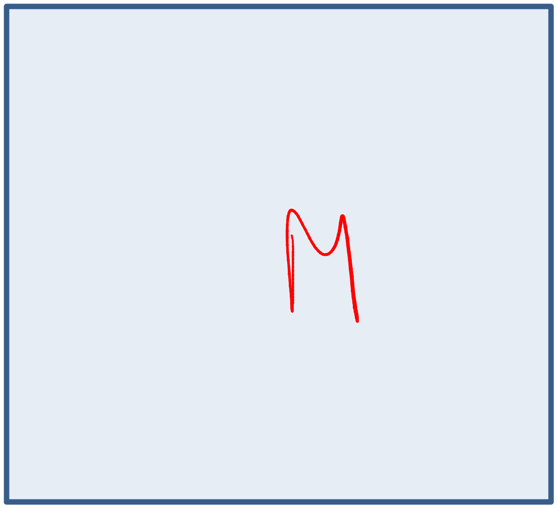
off-diagonal $|A \oplus B| = 2$

$$\frac{1}{(k+1)^2} \cdot \frac{\pi_{k+1}(A \cup B)}{\pi_k(A)}$$

diagonal

$$\frac{1}{k+1} = \sum_{a \notin A} \frac{\pi_{k+1}(A \cup \{a\})}{\pi_k(A)}$$

$\mathrm{diag}(\pi_k) D_k U_{k-1}$



$A = W \cup \{a\}$

$B = W \cup \{b\}$

$$\sum_{W} \pi_{k-1}(W) \sum_{a,b \notin W} f(W \cup \{a\})\, f(W \cup \{b\})$$

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)}$$

off-diagonal $|A \oplus B| = 2$

$$\frac{1}{k^2} \cdot \frac{\pi_k(A)\, \pi_k(B)}{\pi_{k-1}(A \cap B)}$$

$$\sum_{A,B} f(A) f(B) \cdot M_{A,B} =$$

$$= \sum_{W} \pi_{k-1}(W) \sum_{a,b} \frac{M_{A,B}}{\pi_{k-1}(W)} f(A) f(B)$$

diagonal

$$\frac{1}{k^2} \cdot \sum_{a \in A} \frac{\pi_k(A) \pi_k(A)}{\pi_{k-1}(A \setminus \{a\})}$$

$U_k D_{k+1}$

off-diagonal $|A \oplus B| = 2$

$$\frac{1}{(k+1)^2} \cdot \frac{\pi_{k+1}(A \cup B)}{\pi_k(A)}$$

diagonal

$$\frac{1}{k+1} = \sum_{a \notin A} \frac{\pi_{k+1}(A \cup \{a\})}{\pi_k(A)}$$

$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I$$

off-diagonal $|A \oplus B| = 2$

$$\frac{1}{k(k+1)} \cdot \frac{\pi_{k+1}(A \cup B)}{\pi_k(A)}$$

diagonal

$$0 = \frac{1}{k+1} \cdot \frac{k+1}{k} - \frac{1}{k}$$

$$\mathrm{diag}(\pi_k)\left(\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I\right)$$

$$\sum_W \pi_{k-1}(W) \sum_{a,b \notin W} f(W \cup \{a\})\, f(W \cup \{b\})$$

$$\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)}$$

off-diagonal $\quad |A \oplus B| = 2$

$$\frac{1}{k(k+1)} \cdot \pi_{k+1}(A \cup B)$$

diagonal

$$0 = \frac{1}{k+1} \cdot \frac{k+1}{k} - \frac{1}{k}$$

$$f^T \mathrm{diag}(\pi_k)\left(\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \; - D_k U_{k-1}\right) f$$

$$\sum_W \pi_{k-1}(W) \sum_{a,b \notin W} f(W \cup \{a\}) f(W \cup \{b\})$$
$$\left(\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)}\right)$$

$D_k U_{k-1}$

$$\sum_W \pi_{k-1}(W) \sum_{a,b \in W} f(W \cup \{a\})\, f(W \cup \{b\})$$
$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)}$$

off-diagonal $|A \oplus B| = 2$

$$\frac{1}{k^2} \cdot \frac{\pi_k(B)}{\pi_{k-1}(A \cap B)}$$

diagonal

$$\frac{1}{k^2} \cdot \sum_{a \in A} \frac{\pi_k(A)}{\pi_{k-1}(A \setminus \{a\})}$$

$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I$

$$\sum_W \pi_{k-1}(W) \sum_{a,b \in W} f(W \cup \{a\})\, f(W \cup \{b\})$$
$$\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)}$$

off-diagonal $|A \oplus B| = 2$

$$\frac{1}{k(k+1)} \cdot \frac{\pi_{k+1}(A \cup B)}{\pi_k(A)}$$

diagonal

$$0 = \frac{1}{k+1} \cdot \frac{k+1}{k} - \frac{1}{k}$$

2) consider the chain on sets $W \cup \{a\}$
with up-down transition (no self loops)

$$P(a,b) = \frac{\pi_{k+1}(W \cup \{a,b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

IDEA #2: understand $(U_k D_{k+1})$ vs $(D_k U_{k-1})$

1) fix set W

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2}) D^{1/2}(P - 1^T\pi) D^{-1/2}(D^{1/2} f)$$

$$\sum_{a,b \in W} f(W \cup \{a\}) f(W \cup \{b\}) \left(\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)}\right)$$

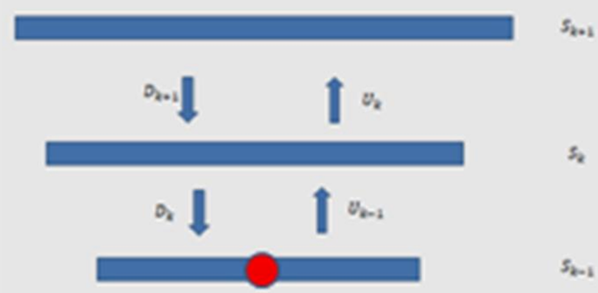
$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_a f(W \cup \{a\})^2 \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

$$f^T Df = (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_S f(S)^2 \pi_k(S) = \sum_W \pi_{k-1}(W) \sum_{a \notin W} f(W \cup \{a\})^2 \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1(W)}}$$

IDEA #2: understand $(U_k D_{k+1})$ vs $(D_k U_{k-1})$



1) fix set W

2) consider the chain on sets $W \cup \{a\}$
with up-down transition (no self loops)

$$P(a,b) = \frac{\pi_{k+1}(W \cup \{a,b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2}) D^{1/2}(P - 1^T\pi) D^{-1/2}(D^{1/2} f)$$

$$\sum_{a,b \notin W} f(W \cup \{a\}) f(W \cup \{b\}) \left(\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)} \right)$$

$$f^T Df = (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_a f(W \cup \{a\})^2 \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

Assume that all local walks have $\lambda_2 \leq \gamma < 1$

IDEA #2: understand $(U_k D_{k+1})$ vs $(D_k U_{k-1})$

1) fix set W

2) consider the chain on sets $W \cup \{a\}$
with up-down transition (no self loops)

$$P(a,b) = \frac{\pi_{k+1}(W \cup \{a,b\})}{(k+1)\pi_k(W \cup \{a\})}$$

Stationary distribution

$$\frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

$$f^T(DP - \pi\pi^T)f = (f^T D^{1/2}) D^{1/2}(P - 1^T\pi)D^{-1/2}(D^{1/2} f)$$

$$\sum_{a,b \in W} f(W \cup \{a\}) f(W \cup \{b\}) \left(\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)} \right)$$

$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_a f(W \cup \{a\})^2 \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)}$$

$$\frac{A_1}{B_1} \leq \gamma$$

$$\frac{A_2}{B_2} \leq \gamma$$

$$\vdots$$

$$\frac{A_n}{B_n} \leq \gamma$$

$$\Rightarrow \quad \frac{c_1 A_1 + c_2 A_2 + \cdots + c_n A_n}{c_1 B_1 + c_2 B_2 + \cdots + c_n B_n} \leq \gamma$$

$$f^T \mathrm{diag}(\pi_k)\left(\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \ - D_k U_{k-1}\right)f$$

$$\sum_W \pi_{k-1}(W) \sum_{a,b \notin W} f(W \cup \{a\}) f(W \cup \{b\}) \left(\frac{\pi_{k+1}(W \cup \{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1}(W)} \cdot \frac{\pi_k(W \cup \{b\})}{k\pi_{k-1}(W)}\right)$$

$$f^T D f = (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_S f(S)^2 \pi_k(S) = \sum_W \pi_{k-1}(W) \sum_{a \notin W} f(W \cup \{a\})^2 \frac{\pi_k(W \cup \{a\})}{k\pi_{k-1(W)}}$$

We have shown (Kauffman, Oppenheim'18)

$$f^T \mathrm{diag}(\pi_k)\left(\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \ - D_k U_{k-1}\right)f \leq \gamma\, f^T \mathrm{diag}(\pi_k) f$$

$$\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \ - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma I$$

We have shown (Kauffman, Oppenheim'18):

$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I \; - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma I$$

$$I - U_k D_{k+1} \; \succcurlyeq_{\pi_k} \frac{k}{k+1} (I - D_k U_{k-1}) - \gamma I$$

If $\gamma \leq 0$ then $1 - \lambda_2(D_r U_{r-1}) \geq \frac{1}{r+1}$

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Can efficiently sample bases of a matroid.

$$f^T \mathrm{diag}(\pi_k)\left(\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \;- D_k U_{k-1}\right) f$$

$$\sum_W \pi_{k-1}(W) \sum_{a,b\notin W} f(W\cup\{a\}) f(W\cup\{b\}) \left(\frac{\pi_{k+1}(W\cup\{a,b\})}{k(k+1)\pi_{k-1}(W)} - \frac{\pi_k(W\cup\{a\})}{k\pi_{k-1}(W)}\cdot\frac{\pi_k(W\cup\{b\})}{k\pi_{k-1}(W)}\right)$$

$$f^T D f \;= (f^T D^{1/2})(D^{1/2} f)$$

$$\sum_S f(S)^2 \pi_k(S) = \sum_W \pi_{k-1}(W) \sum_{a\notin W} f(W\cup\{a\})^2 \frac{\pi_k(W\cup\{a\})}{k\pi_{k-1(W)}}$$

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$$f^T \mathrm{diag}(\pi_k)\left(\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \;- D_k U_{k-1}\right) f \le \gamma\, f^T \mathrm{diag}(\pi_k)(I - D_k U_{k-1}) f$$

$$\frac{k+1}{k}U_k D_{k+1} - \frac{1}{k}I \;- D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma(I - D_k U_{k-1})$$

Alev, Lau'20

$$\frac{k+1}{k} U_k D_{k+1} - \frac{1}{k} I \ - D_k U_{k-1} \preccurlyeq_{\pi_k} \gamma(I - D_k U_{k-1})$$

$$I - U_k D_{k+1} \ \succcurlyeq_{\pi_k} (1-\gamma) \frac{k}{k+1} (I - D_k U_{k-1})$$

If $\gamma_k \leq \frac{C}{n-k}$ then $1 - \lambda_2(D_r U_{r-1}) \geq r^{-O(C)}$

Anari, Liu, Oveis Gharan'20

Can efficiently sample from antiferromagnetic 2-spin models in uniqueness.