Partition Functions: Zeros and efficient approximation IV

Viresh Patel and Guus Regts

Summerschool on Algorithms, Dynamics, and Information Flow in Networks, Dortmund

June 27–July 1, 2022

Introduction

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Theorem (de Boer, Buys, Guerini, Peters, R. 2021+)

Let $\Delta \geq 3$. The closure of $\mathcal{Z}_\Delta$ is contained in the closure of $\mathcal{P}_\Delta$.

Short recap of yesterday's talk

$$Z_{G(T)}(\mu) \cong Z_{T-u}(\mu)\left(Z_{G-v}(\mu) + y Z_{G\setminus N[v]}(\mu)\right).$$

Assumption on μ

(Assumption)

Let $\Delta \geq 4$. Assume that μ is such that on input of any $y \in \mathbb{Q}[i]$ and $\varepsilon \in (0,1)$ we can compute in time $\text{poly}(\log(1/\varepsilon) + \text{size}(y))$ a rooted tree $(T,u) \in \mathcal{G}_\Delta$ such that $\deg_T(u) = 1$ and

- 1 $|R_{T,u}(\mu) - y| \leq \varepsilon$,
- 2 $|T| = \text{poly}(\log(1/\varepsilon) + \text{size}(y))$,
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We want to show that

$$\overline{\mathcal{Z}_\Delta} \subseteq \overline{\{\mu \in \mathbb{Q}[i] \mid \mu \text{ satisfies assumptions 1–3}\}}$$

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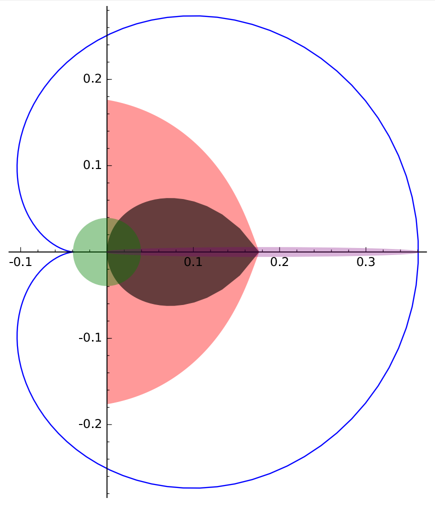
Instead we will show

$$\overline{\mathcal{Z}_\Delta} \subseteq \overline{\{\mu \in \mathbb{Q}[i] \mid \mu \text{ satisfies assumptions 1 and 3}\}}$$

The independence polynomial on ℂ



Addressing • 3

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Lemma

The set $\mathcal{E}_\Delta$ is finite.

Addressing • 1: relating zeros with ratios

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Lemma

Let $\lambda \in \mathbb{C}$ such that there exists a graph $G \in \mathcal{G}_\Delta$ such that $Z_G(\lambda) = 0$. Then there exists a graph $H \in \mathcal{G}_\Delta$ such that $Z_H(\lambda) = 0$ and $R_{H,v} = -1$ for each $v \in V(H)$.

Zeros of graphs are zeros of trees

Lemma (Bencs, 2018)

Let $G \in \mathcal{G}_\Delta$. Then there exists a tree $T \in \mathcal{G}_\Delta$ such that $Z_G | Z_T$. In particular all zeros of Z_G are zeros of Z_T.

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Corollary

Let $\lambda \in \mathbb{C}$ such that there exists a graph $G \in \mathcal{G}_\Delta$ such that $Z_G(\lambda) = 0$. Then there exists a graph $T \in \mathcal{G}_\Delta$ such that $Z_T(\lambda) = 0$ and $R_{T,v} = -1$ for a vertex $v \in V(T)$ of degree 1.

Paths and Möbius transformations

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$$R_{\hat{P}_n,v_n} = (f_{\mu_n} \circ \cdots \circ f_{\mu_1})(0)$$

Intermezzo on Möbius transformations

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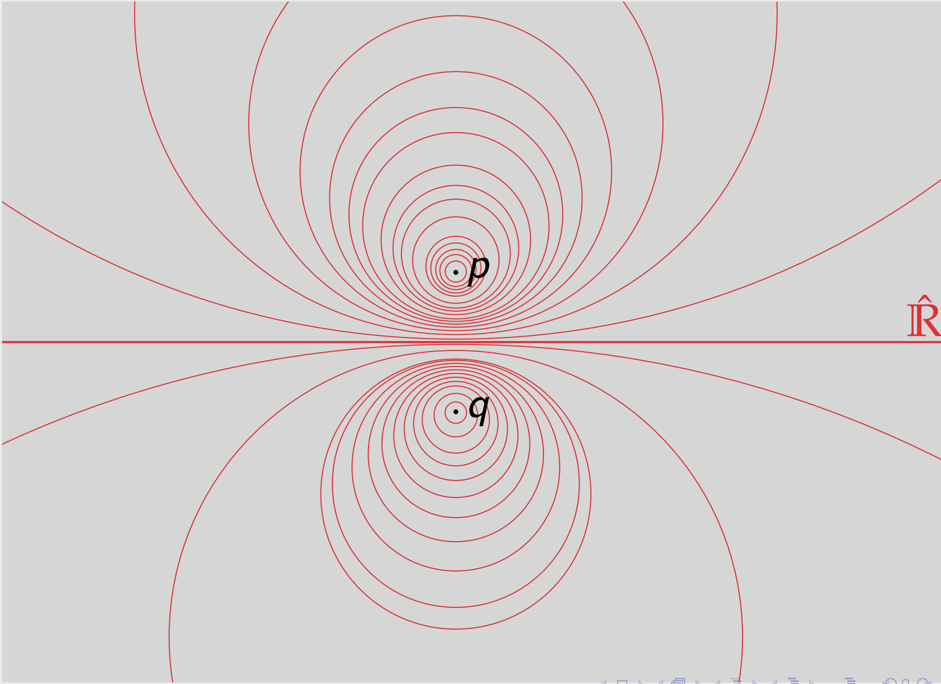
- For an irrational parameter λ, the complex plane $\mathbb{C}$ is foliated with generalized circles on which f_λ acts conjugately to an irrational rotation.


p
q
$\hat{\mathbb{R}}$

Implementations

Suppose now that $\mu \in \mathcal{Z}_\Delta$. We want to show that for some μ' near μ the set

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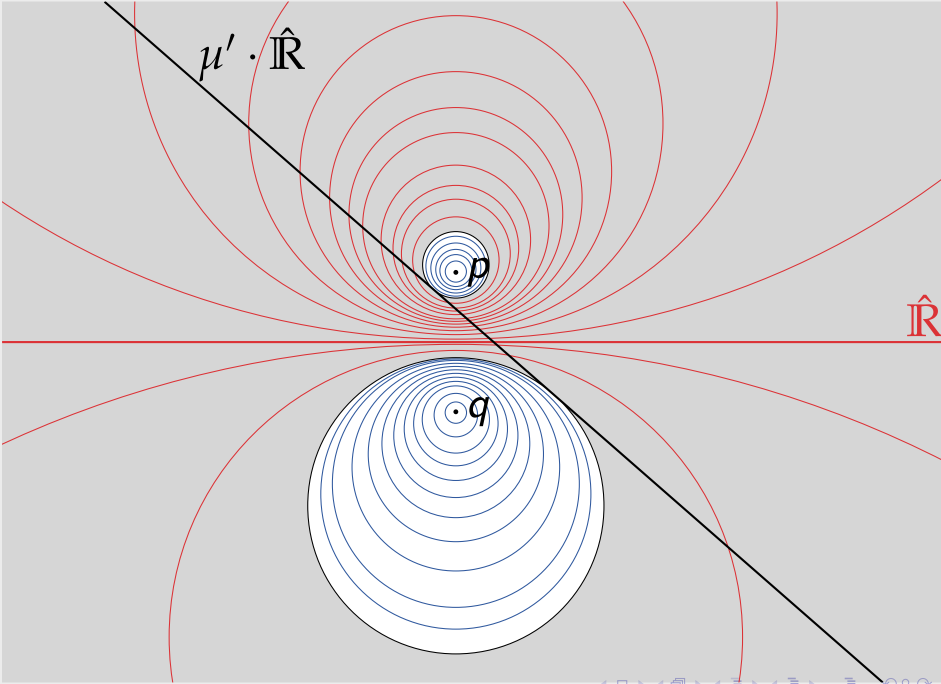
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We showed:
The closure of $\mathcal{Z}_\Delta$ is contained in the closure of $\mathcal{D}_\Delta$. This is in fact an equality!

Addressing • 2

(Assumption)

Let $\Delta \geq 4$. Assume that μ is such that on input of any $y \in \mathbb{Q}[i]$ and $\varepsilon \in (0,1)$ we can compute in time $\text{poly}(\log(1/\varepsilon) + \text{size}(y))$ a rooted tree $(T, u) \in \mathcal{G}_\Delta$ such that $\deg_T(u) = 1$ and

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Addressing the algorithmic part and • 2 relies on properties of Möbius transformations and is quite general.

Thank you for your attention!