Partition Functions: Zeros and efficient approximation III

Viresh Patel and Guus Regts

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Introduction

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Absence of zeros for partition functions $\Rightarrow$ efficient approximation algorithms.

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This and the next lecture:
What about presence of zeros?

The matching polynomial

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Theorem (Heilmann and Lieb, 1972)

Let $\Delta \geq 3$. Then for any graph $G \in \mathcal{G}_\Delta$, and any $z \notin (-\infty, -\frac{1}{4(\Delta-1)})$, $M_G(z) \neq 0$ and this is tight.

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Theorem (Bezáková, Galanis, Goldberg Štefankovič, 2021)

Let $\Delta \geq 3$ and let $z < -\frac{1}{4(\Delta-1)}$ be rational. Approximating the absolute value of $M_G(z)$ for $G \in \mathcal{G}_\Delta$ is #P-hard.

The partition function of the ferromagnetic Ising model

The *partition function of the ferromagnetic Ising model* of a graph $G = (V, E)$ is defined as

$$Z_G(\lambda, b) = \sum_{S \subseteq V} \lambda^{|S|} b^{e(S, V \setminus S)}$$

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Theorem (Peters, R., 2020)

Let $b \in (0, 1)$. Then there exists $\Delta_b > 0$ such that the roots $Z_G(\lambda, b)$ as a polynomial in λ as G ranges over $\mathcal{G}_{\Delta_b}$ are dense in the unit circle.

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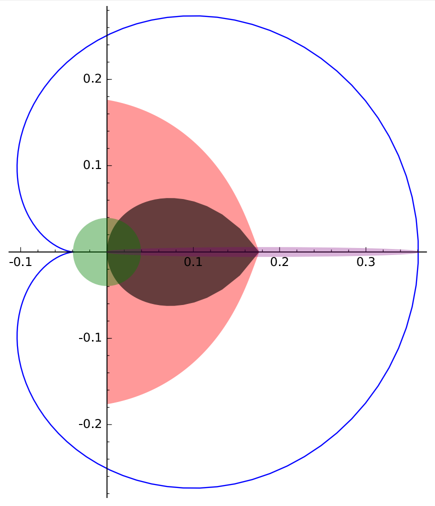
Theorem (Buys, Galanis, Patel, R., 2022)

Let $b \in (0, 1) \cap \mathbb{Q}$. Let $\lambda \in \mathbb{Q}[i] \setminus \mathbb{R}$ such that $|\lambda| = 1$. Approximating the absolute value of $Z_G(\lambda, b)$ for $G \in \mathcal{G}_{\Delta_b}$ is #P-hard.

The independence polynomial on $\mathbb{C}$



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$$\mathcal{Z}_\Delta := \{\lambda \in \mathbb{C} \mid Z(G;\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}$$
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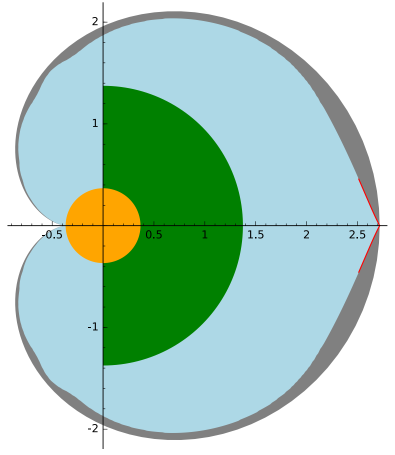
Theorem (de Boer, Buys, Guerini, Peters, R. 2021+)

Let $\Delta \geq 3$. The closure of $\mathcal{Z}_\Delta$ is contained in the closure of $\mathcal{P}_\Delta$.

State of the art for the independence polynomial as $\Delta \to \infty$



Overview of the rest of the lectures

Ingredients for 'zeros implies hardness' for the independence polynomial on $\mathbb{C}$.

- Why is approximating as hard as exact computing?
- What do the complex zeros have to do with this?

Precise formulation of the problem

Let $\lambda \in \mathbb{Q}[i]$ and $\Delta \geq 4$.

Name #Hardcorenorm(λ, Δ).

Instance A graph $G \in \mathcal{G}_\Delta$.

Output If $Z_G(\lambda) = 0$ the algorithm may output any rational number; otherwise the algorithm must output a rational number N such that

$$\frac{1}{1.001} \leq |Z_G(\lambda)|/N \leq 1.001.$$

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We will show that for certain λ's, having acces to an algorithm that solves $\#\text{Hardcorenorm}(\lambda, \Delta)$ in polynomial time, we can compute $Z_G(\lambda)$ exactly in time polynomial in $|V(G)|$ for graphs $G \in \mathcal{G}_3$.

Reducing approximate counting to exact counting: ratios I

Let $\mu \in \mathbb{Q}[i]$ and $H \in \mathcal{G}_3$. We want to compute $Z_H(\mu)$ exactly.

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Lemma

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- *if $Z_{G \setminus N[v]}(\mu) \neq 0$, then $b = 1$ and $r = \frac{Z_G(\mu)}{Z_{G\setminus N[v]}(\mu)}$*

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- *if $Z_{G\setminus N[v]}(\mu) = 0$ and $Z_{G-v}(\mu) \neq 0$, then $b = 0$ and $r = \frac{Z_G(\mu)}{Z_{G-v}(\mu)}$.*

Then $Z_G(\mu)$ can be computed in time polynomial in $|V(G)|$.

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$$\lambda \mapsto \frac{Z_G^{v\ \mathrm{in}}(\lambda)}{Z_G^{v\ \mathrm{out}}(\lambda)} = \frac{\lambda Z_{G \setminus N[v]}(\lambda)}{Z_{G-v}(\lambda)}.$$

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It thus suffices to be able to compute $R_{G,v}$ in the previous lemma.

An assumption on μ

(Assumption)

Let $\Delta \geq 4$. Assume that μ is such that on input of any $y \in \mathbb{Q}[i]$ and $\varepsilon \in (0,1)$ we can compute in time $\text{poly}(\log(1/\varepsilon) + \text{size}(y))$ a rooted tree $(T, u) \in \mathcal{G}_\Delta$ such that $\deg_T(u) = 1$ and

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Given $(G,v) \in \mathcal{G}_3$ and (T,u) as above make a new graph $G(T) \in \mathcal{G}_\Delta$ by identifying u with v.

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$$Z_{G(T)}(\mu) \cong Z_{T-u}(\mu)\left(Z_{G-v}(\mu) + yZ_{G\setminus N[v]}(\mu)\right).$$

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Using an algorithm for #Hardcorenorm(μ, Δ) we can compute an η-approximation $\hat{f}(y)$ to $|f(y)|$ (recall $\eta = 1.001$)

Intermezzo on rational numbers

Let $\lambda = x + iy \in \mathbb{Q}[i]$ be fixed. Let $G = (V, E)$ be an n-vertex graph. Then

- $|Z_G(\lambda)| \leq 2^{O(n)}$,
- the bit size of the number $Z_G(\lambda)$ is $O(n)$,
- if $Z_G(\lambda) \neq 0$, then $|Z_G(\lambda)| > 2^{-O(n)}$.

Reducing exact counting approximate counting: binary search II

Assume $A \neq 0$. Denote $y^* = -B/A$. Note that since $\mu \in \mathbb{Q}[i]$ we must have that y^* is contained in some square box S of diameter $D = 2^{O(|V(G)|)}$ with center m.

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Let $y_1 = m - 4\eta D \quad y_2 = m + 4\eta D$. Then

$$\hat{f}(y_1) - \hat{f}(y_2) \geq |A| \left(|y_1 - y^*| - |y_2 - y^*| - 2\eta D\right)$$

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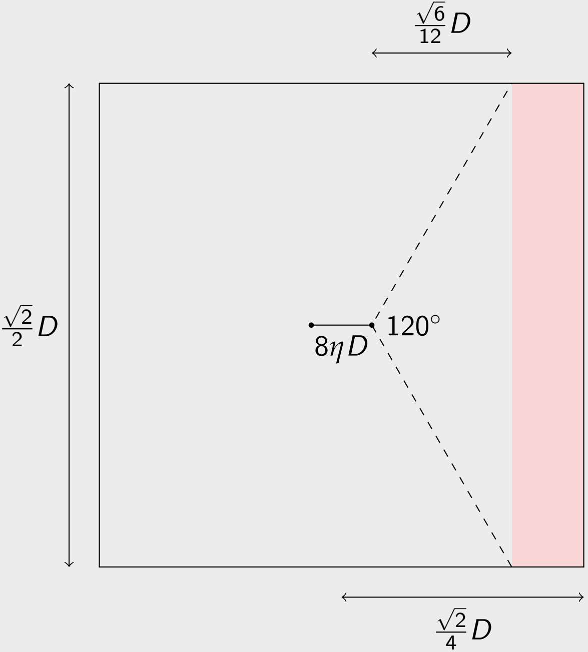
Suppose now that $\hat{f}(y_1) \leq \hat{f}(y_2)$. Then y^* cannot be contained in the cone of 120° centered at y_2 pointing towards y_1.


$\frac{\sqrt{6}}{12}D$
$\frac{\sqrt{2}}{2}D$
$8\eta D$
120°
$\frac{\sqrt{2}}{4}D$

Box shrinking

After $O(|V(G)|)$ many steps the initial box S shrinks down to a box S' of diameter

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- If $A = 0$ and $B \neq 0$, then for any $y \in S'$, $\hat{f}(y) \geq (1+\eta)|B|$.

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- If $A \neq 0$ (and thus y^* is contained in the initial box), we can determine y^* exactly (being the unique complex number with rational coordinates whose denominators are bounded by $2^{O(|V(G)|)}$.)

Conclusion

Since $\varepsilon = 2^{-O(|V(G)|)}$, having an poly-time algorithm for $\#$Hardcorenorm(μ, Δ), the box shrinking procedure gives us a polynomial time algorithm to detect whether $A = 0$ or not and compute $1/y^* = -A/B$ provided not both A and B are equal to 0.

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(Summary)

- Use 'box shrinking' to compute ratios $R_{G,v}$ exactly.
- Use ratios to compute $Z_H(\mu)$ exactly with telescoping product.
- Be careful when and where to 'trust' the algorithm.

Thank you for your attention!