Partition functions: complex zeros and algorithms

Part 2

Viresh Patel and Guus Regts

Summer school on algorithms, dynamics, and information flow in networks

Recap

Taylor Polynomial Interpolation Method (Barvinok)

- Let $p = p_G$ be a (graph) polynomial of degree $\leq n$.
- Assume $p(z) \neq 0$ for all $|z| \leq R$ for some $R > 0$. ($z \in \mathbb{C}$)
- Let $f(z) = \ln p(z)$ for $|z| < R$ and let

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Recipe for FPTAS

- Identify zero-free region of p containing z (inc. non-disks).
- Efficiently compute $f^{(k)}(0)$ for $k = 0, \ldots, O(\ln n/\varepsilon)$.

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$$\lambda^*(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta} \qquad \lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta} \qquad (\text{Note } \lambda^* < \lambda_c)$$

Theorem

We have $Z_G(z) \neq 0$ for all $z \in D$ and $\Delta(G) \leq \Delta$ where

(1) $D = \{z \ : \ |z| \leq \lambda^*\}$ *(Dobushin, Shearer)*

(2) $D =$ *open region containing* $[0, \lambda_c)$ *(Peters, Regts)*

(3) $D = \{z : \Re(z) \geq 0, \ |z| \leq \frac{7}{8} \tan\left(\frac{\pi}{2(\Delta-1)}\right)\}$ *(Csikvári, Bencs)*

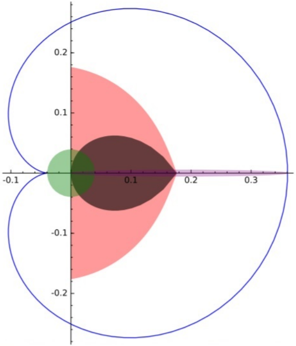
Theorem (Dobrushin, Shearer 1990's)

For $G = (V, E)$ with $\Delta(G) \leq \Delta$

$$|\lambda| \leq \lambda^*(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta} \implies Z_G(\lambda) \neq 0.$$

Remark Here $\lambda^*(\Delta)$ is best possible since

$$\text{Zeros of } Z_{T_{\Delta,r}} \text{ converge to } -\lambda^*(\Delta).$$



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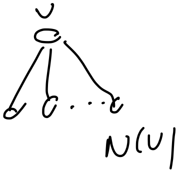
$$Z_G(\lambda) = Z_{G-v}(\lambda) \;+\; \lambda Z_{G-N[v]}(\lambda)$$

$$\frac{Z_G(\lambda)}{Z_{G-v}(\lambda)} = \quad 1 \qquad + \lambda\frac{Z_{G-N[v]}(\lambda)}{Z_{G-v}(\lambda)} =: 1 + R_{G,v} \quad \text{if } Z_{G-v}(\lambda) \neq 0.$$

v fixed

$$Z_G(\lambda) = \sum_{\substack{S \subseteq V \\ \text{ind}}} \lambda^{|S|} = \sum_{\substack{S:\, v \notin S \\ S \text{ ind}}} \lambda^{|S|} + \sum_{\substack{S:\, v \in S \\ S \text{ ind}}} \lambda^{|S|}$$



$$= Z_{G-v}(\lambda) + \lambda Z_{G-N[v]}(\lambda)$$

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$v_0\in V$, $N(v_0)=\{v_1,\dots v_d\}$

$$R_{G,v_0} = \lambda\prod_{i=1}^{d}\left(1+R_{G-\{v_0,\dots,v_{i-1}\},v_i}\right)^{-1}$$

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$$R_{G,v_0} := \lambda \frac{Z_{G-N[v_0]}}{Z_{G-v_0}}$$

$$\frac{Z_{G-v_0}}{Z_{G-N[v_0]}} = \frac{Z_{G-v_0}}{Z_{G-\{v_0,v_1\}}} \cdot \frac{Z_{G-\{v_0,v_1\}}}{Z_{G-\{v_0,v_1,v_2\}}} \cdots \frac{Z_{G-\{v_0,\dots,v_{d-1}\}}}{Z_{G-\{v_0,\dots,v_d\}}}$$

$$\| $$

$$\frac{\lambda}{R_{G,v_0}} = (1 + R_{G-v_0,v_1})(1 + R_{G-\{v_0,v_1\},v_2}) \cdots (1 + R_{G-\{v_0,\dots,v_{d-1}\},v_d})$$

Then rearrange.

Theorem (Dobrushin, Shearer 1990's)

For $G = (V, E)$ *with* $\Delta(G) \leq \Delta$,
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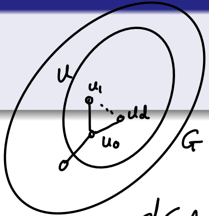

$d \leq \Delta - 1$

Proof: Assume G connected and fix $v \in V$.

Claim: For all $U \subseteq V - \{v\}$, we have

(i) $Z_{G[U]}(\lambda) \neq 0$

(ii) If $u_0 \in U$ has a neighbour outside U then $|R_{G[U],u_0}| < 1/\Delta$

Induction on $|U|$. $|U| = 0$ $\quad Z_{G[U]}(\lambda) = 1 \neq 0$

(ii) vacuous. Base case ✓

$|U| \geq 1$ $\quad |R_{G[U],u_0}| = |\lambda| \prod_{i=1}^{d} |(1 + R_{G[U]-\{u_0,\dots,u_{i-1}\},u_i})|^{-1}$

$< \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}} \left(1 - \frac{1}{\Delta}\right)^{-(\Delta-1)} = \frac{1}{\Delta}$ $\quad$ proves (ii)

(i) holds since $R \neq -1$ by (ii)

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$Z_G(\lambda)\neq 0$ iff $R_{G,v}\neq -1$ (and $Z_{G-v}(\lambda)\neq 0$).

$v_0\in V,\quad N(v_0)=\{v_1,\dots v_d\}$

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$d \leq \Delta$

$$|R_{G,v}| = |\lambda| \prod_{i=1}^{d} \left|\left(1 + R_{G-\{v,v_1,\dots,v_{i-1}\},v_i}\right)\right|^{-1}$$

$$< \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}} \left(1 - \frac{1}{\Delta}\right)^{-\Delta} = \frac{1}{\Delta-1}$$

So $R_{G,v} \neq -1 \Rightarrow Z_G(\lambda) \neq 0.$

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Fix graph $G = (V, E)$ and parameter $b \in (0, 1)$

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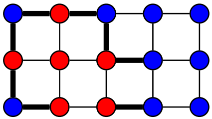

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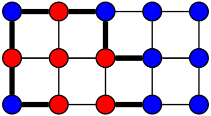

$$\mathbb{P}(\sigma) \propto b^{\delta(\sigma)}$$

$$\text{i.e.} \quad \mathbb{P}(\sigma) = \frac{b^{\delta(\sigma)}}{Z_G(b)} \quad \text{where} \quad Z_G(b) = \sum_{\sigma : V \to \{+.-\}} b^{\delta(\sigma)}.$$

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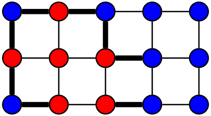

$$\mathbb{P}(\sigma) \propto \lambda^{n_+(\sigma)} b^{\delta(\sigma)}$$

i.e. $\mathbb{P}(\sigma) = \dfrac{\lambda^{n_+(\sigma)} b^{\delta(\sigma)}}{Z_G(b, \lambda)}$ where $Z_G(b, \lambda) = \displaystyle\sum_{\sigma : V \to \{+.-\}} \lambda^{n_+(\sigma)} b^{\delta(\sigma)}$.

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We have $\mathrm{Cut}(\mathbf{z}, B) \neq 0$ *whenever* $B \in \mathbb{R}^{n \times n}$ *is symmetric with* $B_{ij} \in [-1, 1]\ \forall i, j$, *and* $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ *with* $|z_i| < 1\ \forall i$.

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Remark Same holds if we replace "$|z_i| < 1$ $\forall i$" with "$|z_i| > 1$ $\forall i$"

Theorem (Lee-Yang (univariate))

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Extensions to hypergraphs (Liu, Sinclair, Srivastava)

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Problem Can we remove the condition $\Delta(G) \leq \Delta$?

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$p \in \mathbb{C}[z_1, \ldots, z_n]$ is $\mathbb{D}$-stable if $z_i \in \mathbb{D}\ \forall i \implies p(z_1, \ldots, z_n) \neq 0$.

$\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$.

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Write $z^S = \prod_{i \in S} z_i$. For $f, g \in \mathbb{C}[z_1, \ldots, z_n]$ of the form

$$f \equiv \sum_{S \subseteq [n]} a_S z^S, \quad g \equiv \sum_{S \subseteq [n]} b_S z^S$$

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$$f \equiv \sum_{S \subseteq [n]} a_S z^S, \quad g \equiv \sum_{S \subseteq [n]} b_S z^S \quad \text{define} \quad f * g := \sum_{S \subseteq [n]} a_S b_S z^S$$

$$\mathrm{Cut}(\mathbf{z}, B) = \sum_{S \subseteq V} \prod_{i \in S,\, j \notin S} B_{ij} \prod_{i \in S} z_i$$

Theorem (Lee-Yang 1953)

We have $\mathrm{Cut}(\mathbf{z}, B) \neq 0$ *whenever* $B \in \mathbb{R}^{n \times n}$ *is symmetric with* $B_{ij} \in [-1, 1]\ \forall i, j$*, and* $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ *with* $|z_i| < 1\ \forall i$*.*

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Pf induction on $n = \#$ variables

For $n=1$ $\quad f = a + bz$ and $g = c + dz$ $\mathbb{D}$-stable

$\Rightarrow \frac{|a|}{|b|} > 1 \qquad \frac{|c|}{|d|} > 1$

$\Rightarrow \frac{|ac|}{|bd|} > 1 \Rightarrow ac + bdz$ is $\mathbb{D}$-stable ✓

Assume $n>1$ and f, g as given. Assume $\mathbb{D}$-stable.

$$f = \sum_{S \subseteq [n-1]} (a_S + a_{S\cup\{n\}} z_n) z^S$$

$$g = \sum_{S \subseteq [n-1]} (b_S + b_{S\cup\{n\}} z_n) z^S$$

For any $u, v \in \mathbb{D}$ let

$$f_u(z_1, \dots, z_{n-1}) = f(z_1, \dots, z_{n-1}, u) \in \mathbb{C}[z_1, \dots, z_{n-1}]$$
$$g_v(z_1, \dots, z_{n-1}) = g(z_1, \dots, z_{n-1}, v) \in \mathbb{C}[z_1, \dots, z_{n-1}]$$

f_u, g_v are $\mathbb{D}$-stable $\Rightarrow$ $f_u * g_v$ is $\mathbb{D}$-stable

$$f_u * g_v = \sum_{S \subseteq [n-1]} (a_S + a_{S\cup\{n\}} u)(b_S + b_{S\cup\{n\}} v) z^S$$

$$f_u * g_v \equiv \sum_{S\subseteq[n-1]} (a_S + a_{S\cup\{n\}} u)(b_S + b_{S\cup\{n\}} v)\, z^S$$

For fixed $z_1,\dots,z_{n-1} \in \mathbb{D}$ write $p(u,v) = f_u * g_v \in \mathbb{C}[u,v]$

p is $\mathbb{D}$-stable since $f_u * g_v \neq 0$ $\forall u, v \in \mathbb{D}$

$$p(u,v) \equiv \sum_{S\subseteq[n]} a_S b_S z^S + \sum_{S\subseteq[n-1]} b_S a_{S\cup\{n\}} z^S u + \sum_{S\subseteq[n-1]} a_S b_{S\cup\{n\}} z^S v + \sum_{S\subseteq[n-1]} a_{S\cup\{n\}} b_{S\cup\{n\}} z^S uv$$

$$P(u,v) = \sum_{S \subseteq [n-1]} a_S b_S z^S + \cancel{\sum_{S \subseteq [n-1]} b_S a_{S \cup \{n\}} z^S u}$$ by prop

$$+ \cancel{\sum_{S \subseteq [n-1]} a_S b_{S \cup \{n\}} z^S v} + \sum_{S \subseteq [n-1]} a_{S \cup \{n\}} b_{S \cup \{n\}} z^S \, uv$$

z_n

$= f * g$ is $\mathbb{D}$-stable as poly in z_n $\forall$ fixed $z_1, \dots, z_{n-1} \in \mathbb{D}$

$\Rightarrow$ $f * g$ is $\mathbb{D}$-stable as poly in $z_1, \dots, z_n$

<u>Prop</u> if $q(x_1, x_2) = a + bx_1 + cx_2 + dx_1 x_2$ is $\mathbb{D}$-stable

$\Rightarrow \hat{q}(x) = a + dx$ is $\mathbb{D}$-stable

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Pf For $i<j$ let $P_{ij} = (1 + B_{ij}(z_i + z_j) + z_i z_j) \sum_{S \subseteq [n] \setminus \{i,j\}} z^S$

$= (1 + B_{ij}(z_i + z_j) + z_i z_j) \prod_{k \in [n] \setminus \{i,j\}} (1 + z_k)$

① $P_{ij} \neq 0$ provided $|z_i| < 1$ $\forall i$

② $\ast_{i<j} P_{ij} = \mathrm{Cut}(B, z)$

① $1 + a z_1 + a z_2 + z_1 z_2 = 0$

$z_2 = -(1 + a z_1)/a + z_1$

This is a Möbius transformation That maps unit circle to itself, but maps inside to outside so if $|z_1|<1$ then $|z_2|>1$.

② coefficient of z^S in P_{ij} $= \begin{cases} B_{ij} & \text{if } i \in S\ j \notin S \\ B_{ij} & \text{if } i \notin S\ j \in S \\ 1 & \text{owise} \end{cases}$

$$\text{coeff of } \mathop{\ast}_{i<j} P_{ij} = \prod_{\substack{i \in S \\ j \notin S}} B_{ij}$$

$$= \text{coeff of } z^S \text{ in } \mathrm{Cut}(B, z)$$

Not in lecture:

Set $q_{ij}^{\rho} = P_{ij}(\rho z_1, \dots, \rho z_n)$ for $\rho \in (0,1)$

q_{ij}^{ρ} is $\mathbb{D}$-stable

Set $q^{\rho} = \mathop{\ast}_{i<j} q_{ij}^{\rho}$ is $\mathbb{D}$-stable.

$q^{\rho} \neq 0$ whenever $|z_i| < 1$ $\forall i$
$\rho \in (0,1)$

$\Rightarrow \lim_{\rho \to 1} q^{\rho} \neq 0$ whenever $|z_i| < 1$ $\forall i$

$\|$ by Hurwitz theorem

$\mathrm{Cut}(B, z)$.

Zeros of q^{ρ} move continuously with ρ.

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Graph $G = (V, E)$.

Call $M \subseteq E$ a matching if no edges in M are incident.

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So our method implies

There is an FPTAS to evaluate $M_G(z)$ with $\Delta(G) \leq \Delta$ and $z \in \mathbb{C} \setminus (-\infty, -\lambda^*(2\Delta - 1))$.

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$\forall u \in V, \quad D_G(z) = zD_{G-\{u\}}(z) - \sum_{v \in N_G(u)} D_{G-\{u,v\}}(z)$

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$$\forall u \in V, \quad D_G(z) = zD_{G-\{u\}}(z) - \sum_{v \in N_G(u)} D_{G-\{u,v\}}(z)$$

Show by induction (on # vertices) that for all $G = (V, E)$

(a) D_G has only real roots;

(b) $\forall u \in V$ if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}\left(\frac{D_G(z)}{D_{G-u}(z)}\right) > 0$.

Solution

All roots of D_G real $\Rightarrow$ All roots of M_G are real and negative.

Suppose M_G has root $x \in \mathbb{C}$

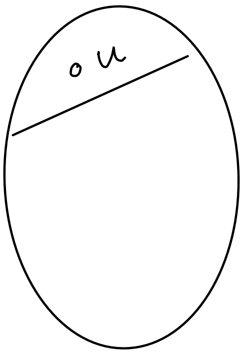
Then D_G has root z where $-z^{-2} = x$

i.e. $z = (-x)^{-1/2}$

Since z is real then $-x$ is positive.

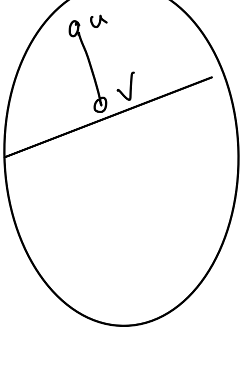
Fix $u \in V$

$$D_G(z) = \sum_{M} (-1)^{|M|} z^{n-2|M|}$$



$$= \sum_{\substack{M \text{ has no} \\ \text{edge incident} \\ \text{to } u}} (-1)^{|M|} z^{n-2|M|} + \sum_{v \in N(u)} \sum_{M : uv \in M} (-1)^{|M|} z^{n-2|M|}$$

$$= z\, D_{G-u}(z) - \sum_{v \in N(u)} D_{G-\{u,v\}}(z)$$



Assume $|G| = n$ and (a), (b) hold for all smaller graphs.

(ii) Fix $u \in V$ and $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$

$$\frac{D_G}{D_{G-u}} = \frac{z D_{G-u} - \sum_{v \in N(u)} D_{G-\{u,v\}}}{D_{G-u}}$$

$$= \underbrace{z}_{\substack{\text{Im part} \\ > 0}} - \sum_{v \in N(u)} \Bigg(\underbrace{\frac{D_{G-u}}{D_{G-\{u,v\}}}}_{\substack{\text{Im part} \\ > 0}} \Bigg)^{-1}$$

by induction

$\Rightarrow \operatorname{Im}\left(\frac{D_G}{D_{G-u}}\right) > 0$ so (b) holds for G

This shows if $\operatorname{Im}(z) > 0$ then $D_G(z) \neq 0$

(Note $D_{G-u}(z) \neq 0$ by induction)

If $\operatorname{Im}(z) < 0$ then $\operatorname{Im}(-z) > 0$ and

$D_G(z) = \pm D_G(-z)$ so $\operatorname{Im}(D_G(z)) \substack{> 0 \\ < 0} \overset{(i)}{\Longrightarrow}$ for G