The rank of adjacency matrices of sparse Erdős-Rényi graphs over finite fields

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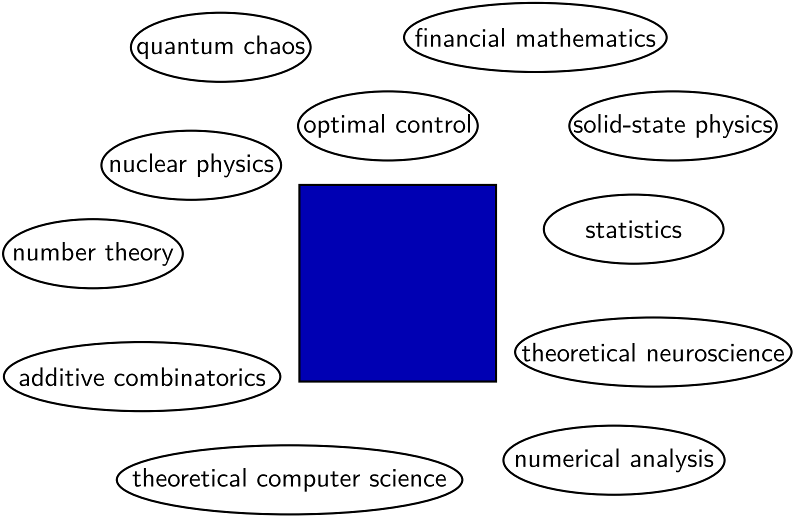
joint work with R. v. d. Hofstad and H. Zhu

Overview

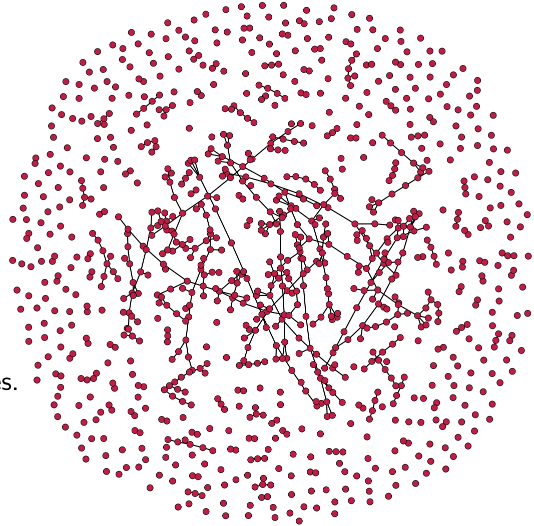
1. Introduction
2. Selected results
3. Rank formula
4. Proof outline

Random matrices



Adjacency matrices

This talk:
Adjacency matrix of
Erdős-Rényi random graph.



Erdős-Rényi random graph $\boldsymbol{G}_{n,p}$:

- $V(\boldsymbol{G}_{n,p}) = \{1, \ldots, n\}$
- $E(\boldsymbol{G}_{n,p})$: Each possible edge is included with probability p, independently of all other edges.

Adjacency Matrix $\boldsymbol{A}_{n,p}$:
$\boldsymbol{A}_{n,p}(i,j) = \mathbb{1}\{\{i,j\} \in E(\boldsymbol{G}_{n,p})\}$

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Question: What can be said about the rank of the matrix $\boldsymbol{A}_{n,p}$?

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This talk: Rank of $\boldsymbol{A}_{n,p}$ over finite field $\mathbb{F}$ in the regime $p = d/n$, $d > 0$.

An easy upper bound

Since every isolated vertex contributes a zero-row to the adjacency matrix, we have

$$\mathrm{rk}_{\mathbb{F}}(\boldsymbol{A}_{n,p}) \leq n - \# \text{ isolated vertices in } \boldsymbol{G}_{n,p}.$$

Moderately sparse setting

Theorem (Costello, Tao & Vu '04, Costello & Vu '06)

Fix $d > 1/2$. Then for any $p \in \left(\frac{d\ln(n)}{n}, \frac{1}{2}\right]$, w.h.p.,

$$\mathrm{rk}_{\mathbb{R}}(\boldsymbol{A}_{n,p}) = n - \#\ \textit{isolated vertices of}\ \boldsymbol{G}_{n,p}.$$

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This implies that

- $\mathrm{rk}_{\mathbb{R}}(\boldsymbol{A}_{n,p}) < n$ w.h.p. for $p \in [0, d\ln(n)/n)$, where $d < 1$, and
- $\mathrm{rk}_{\mathbb{R}}(\boldsymbol{A}_{n,p}) = n$ w.h.p. for $p \in (d\ln(n)/n, 1/2]$, where $d > 1$.

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Assume that $p \leq d \ln(n)/n$ for $d < 1/2$ (and $p = \omega(n^{-3/2})$ is not too small).

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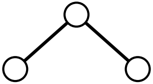
Most obvious obstruction:



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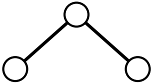

The rows corresponding to the leaves are not zero, but equal, and this further reduces the rank. So w.h.p.,

$$\mathrm{rk}_{\mathbb{R}}(\boldsymbol{A}_{n,p}) < n - \# \text{ isolated vertices of } \boldsymbol{G}_{n,p}.$$

A modified graph

Each cherry increases the nullity by one, and thus the same as an isolated vertex.

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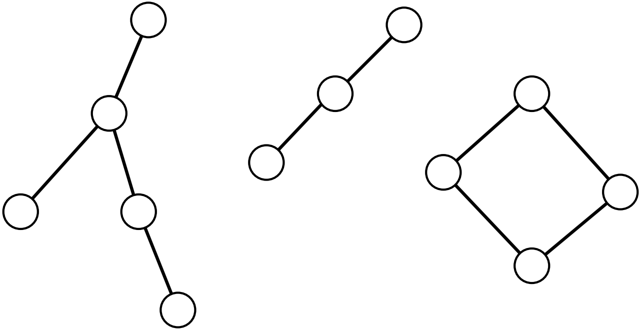
Idea:

Consider an iterative procedure that deletes vertices of degree one and their neighbours
(thus turning cherries into isolated vertices).

Leaf removal process

Procedure:

Iteratively remove vertices of degree one along with their unique neighbour.



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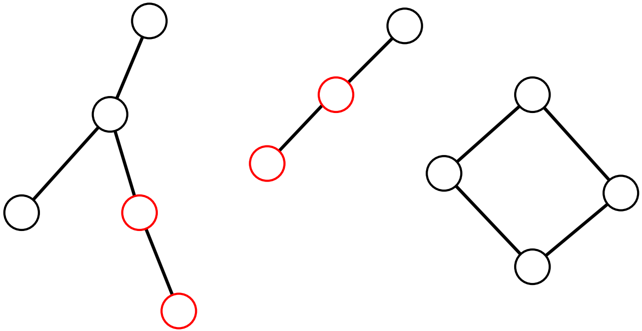

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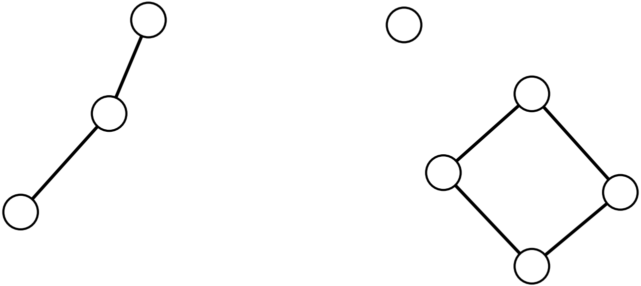

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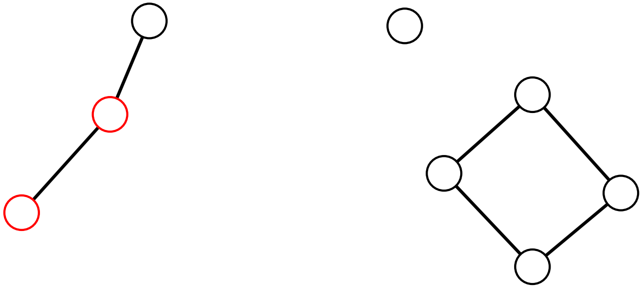

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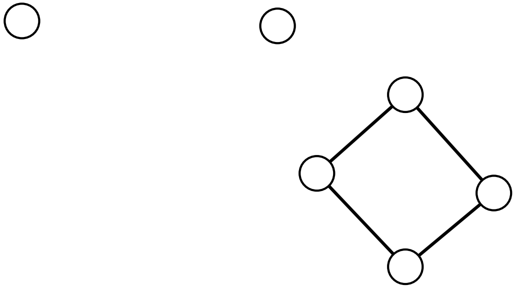

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Observation:

Each step of the process does not change the nullity.

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$$A = \begin{pmatrix} 0 & 1 & \mathbb{0}^T \\ 1 & 0 & b^T \\ \mathbb{0} & b & A' \end{pmatrix}$$

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Then

$$x = (x_1, x_2, x')^T \in \ker(A) \qquad \Longleftrightarrow \qquad \begin{aligned} x_2 &= 0 \\ x_1 &= -b^T x' \\ A'x' &= (-x_2)b \end{aligned}$$

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Conclusion: $\mathrm{nul}(A) = \mathrm{nul}(A')$.

Leaf removal and nullity

Call the graph obtained from $\boldsymbol{G}_{n,p}$ through the leaf removal process $\boldsymbol{G}_{n,p}^{\mathsf{KS}}$ and its adjacency matrix $\boldsymbol{A}_{n,p}^{\mathsf{KS}}$.

Then $\boldsymbol{G}_{n,p}^{\mathsf{KS}}$ has at least as many isolated vertices as the original graph, and no cherries.

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What is the number of isolated vertices in $\boldsymbol{G}_{n,p}^{\mathsf{KS}}$?

Sparse setting

From now on: $p = d/n$ for $d > 0$.

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Let

$$\phi(\alpha) = \exp(d(\alpha - 1))$$

be the probability generating function of a Poisson random variable, and $M_d : [0,1] \to \mathbb{R}$ by

$$M_d(\alpha) = \phi(1 - \phi(\alpha)) + (1 + d(1 - \alpha))\phi(\alpha).$$

Leaf removal and nullity

Theorem (Karp & Sipser (1981), Aronson, Frieze & Pittel (1998))

Fix $d > 0$. Then for $p = d/n$,

$$\frac{\#\ \textit{of isolated vertices in}\ \boldsymbol{G}_{n,p}^{KS}}{n} \xrightarrow{\mathbb{P}} \max_{\alpha \in [0,1]} M_d(\alpha) - 1.$$

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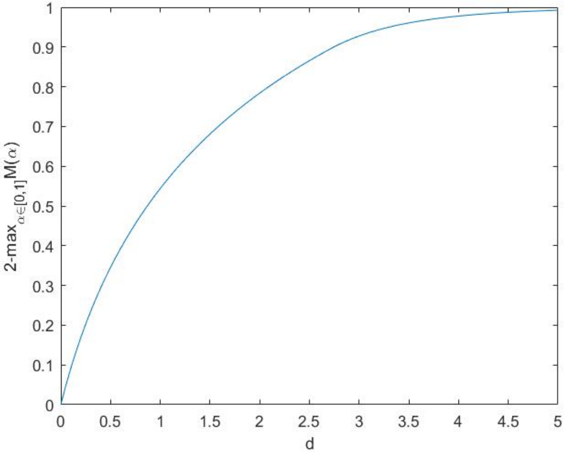
Moreover, for $d \leq e$, the number of non-isolated vertices is $o(n)$ w.h.p. Thus, by the rank-nullity theorem, we obtain the asymptotic rank formula

$$\frac{\mathrm{rk}_{\mathbb{F}}(\boldsymbol{A}_{n,p})}{n} = 1 - \frac{\mathrm{nul}_{\mathbb{F}}(\boldsymbol{A}_{n,p}^{\mathrm{KS}})}{n} \xrightarrow{\mathbb{P}} 2 - \max_{\alpha \in [0,1]} M_d(\alpha).$$

Rank function



Asymptotic rank formula

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Theorem (Bordenave, Lelarge & Salez (2011))

Fix $d > 0$. Then for $p = d/n$,

$$\frac{\mathrm{rk}_{\mathbb{R}}(\boldsymbol{A}_{n,p})}{n} \to 2 - \max_{\alpha \in [0,1]} M_d(\alpha) \qquad a.s.$$

(Bordenave, Lelarge and Salez actually provide an asymptotic rank formula for all random graph sequences $(\boldsymbol{G}'_n)_{n \geq 1}$ converging locally to a rooted Galton-Watson tree, provided the latter satisfies a certain degree condition.)

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Their proof crucially relies on the fact that because of the spectral theorem for symmetric matrices over $\mathbb{R}$, $\boldsymbol{A}_{n,p}$ is diagonalisable by an orthogonal matrix.

(They then analyse the associated spectral measure and provide an explicit formula for the asymptotic multiplicity of the eigenvalue 0 in terms of the degree generating function.)

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(They then analyse the associated spectral measure and provide an explicit formula for the asymptotic multiplicity of the eigenvalue 0 in terms of the degree generating function.)

However, symmetric matrices over finite fields are not necessarily diagonalisable, and the approach through an empirical spectral measure becomes unfeasible.

An upper bound

Since for any field $\mathbb{F}$ and $A \in \{0,1\}^{n\times n}$,

$$\mathrm{rk}_{\mathbb{F}}(A) \le \mathrm{rk}_{\mathbb{Q}}(A) = \mathrm{rk}_{\mathbb{R}}(A),$$

the result of Bordenave, Lelarge and Salez immediately implies the upper bound

$$\limsup_{n\to\infty} \frac{\mathrm{rk}_{\mathbb{F}}(\boldsymbol{A}_n)}{n} \le 2 - \max_{\alpha\in[0,1]} M_d(\alpha) \qquad \text{a.s.}$$

Alternatively, the same upper bound follows from the leaf removal argument and the result of Karp & Sipser.

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Question: Is this bound tight for finite fields and $d > e$?

Towards a more combinatorial approach

A non-symmetric random matrix model

- $(\boldsymbol{k}_i)_{i\geq 1}$ be i.i.d. copies of a random variable $\boldsymbol{k}$ (`ROW`),
- $(\boldsymbol{d}_i)_{i\geq 1}$ be i.i.d. copies of a random variable $\boldsymbol{d}$ (`COLUMN`).

$$
\mathbb{A} = \left(\begin{array}{ccccccccc}
 & & * & & & * & & & \\
* & & & * & & & & * & \\
 & & & & & & & & * \\
 & & & & & & & & \\
 & * & & & & * & & & \\
* & & & & & * & & & * \\
 & & & & & & & & \\
* & & & & * & & & & \\
 & & & * & & & & & \\
 & & & & & & * & * &
\end{array}\right)
\begin{array}{l}
\boldsymbol{k}_1 \\ \boldsymbol{k}_2 \\ \\ \\ \\ \\ \vdots \\ \\ \\ \boldsymbol{k}_m
\end{array}
$$

$$
\begin{array}{ccccccccc}
\boldsymbol{d}_1 & \boldsymbol{d}_2 & & & \cdots & & & & \boldsymbol{d}_n
\end{array}
$$

Asymptotic rank formula for non-symmetric model

Theorem (Coja-Oghlan, Ergür, Gao, Hetterich, Rolvien '22+)

For any field $\mathbb{F}$,

$$\frac{\mathrm{rk}(\mathbb{A})}{n} \xrightarrow{\mathbb{P}} 1 - \max_{z \in [0,1]} \Phi(z) \qquad \text{as } n \to \infty.$$

Rank result for finite fields

Theorem (v. d. Hofstad, M., Zhu '22+)

For any $d > 0$, any coupling $(\boldsymbol{A}_{n,p})_{n\geq 1}$ and any field $\mathbb{F}$,

$$\frac{\mathrm{rk}_{\mathbb{F}}\left(\boldsymbol{A}_{n,p}\right)}{n} \xrightarrow{a.s.} 2 - \max_{\alpha\in[0,1]} M_d(\alpha), \qquad n \to \infty.$$

This recovers the result of Bordenave, Lelarge and Salez for the case $\mathbb{F} = \mathbb{R}$.

Proof Overview

Proof structure

In light of the previous discussion, there are three steps:

1. Upper bound on the expected rank: Via rank result over $\mathbb{R}$ or leaf removal argument.
2. Reduce almost sure lower bound to lower bound on expectations: Via martingale argument.
3. Lower bound on expected rank: Via Aizenman-Sims-Starr scheme.

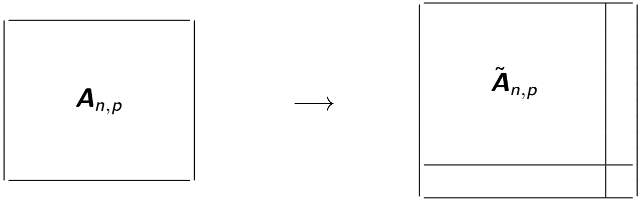
Today: Sketch of third part.

Aizenman-Sims-Starr scheme

Basic idea following Coja-Oghlan, Ergür, Gao, Hetterich, Rolvien:
Grow the matrices one-by-one and observe the change in rank.

By the Stolz-Cesàro Theorem,

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\mathrm{rk}_{\mathbb{F}}(\boldsymbol{A}_{n,p})\right] \geq \liminf_{n\to\infty} \left(\mathbb{E}\left[\mathrm{rk}_{\mathbb{F}}(\boldsymbol{A}_{n+1,p})\right] - \mathbb{E}\left[\mathrm{rk}_{\mathbb{F}}(\boldsymbol{A}_{n,p})\right]\right).$$



Intermediate step

We break this into the two steps

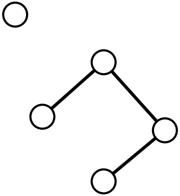
$$\left|\begin{array}{c} \\ \boldsymbol{A}_{n,p} \\ \\ \end{array}\right| \longrightarrow \left|\begin{array}{c} \\ \tilde{\boldsymbol{A}}_{n,p} \\ \\ \end{array}\right|$$

and

$$\left|\begin{array}{c} \\ \tilde{\boldsymbol{A}}_{n,p} \\ \\ \end{array}\right| \longrightarrow \left|\begin{array}{c|c} \tilde{\boldsymbol{A}}_{n,p} & \\ \hline & \\ \end{array}\right|$$

Rank of $\boldsymbol{A}_{n,p}$

The rank may behave in a complex way under the addition of edges:



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{rk}_{\mathbb{F}}(A) = 4$$

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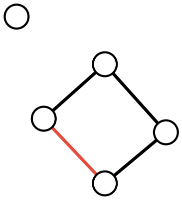

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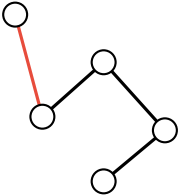

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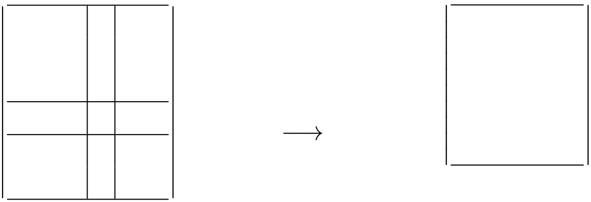
$$\mathrm{rk}_{\mathbb{F}}(A) = 5$$

Symmetric row and column removal

Observation: The core of both operations is symmetric row and column removal.



$\mathsf{rk}_{\mathbb{F}}(A)$ $\longrightarrow$ $\mathsf{rk}_{\mathbb{F}}(A)\langle i; i\rangle$

Frozen variables

The rank increase can be related to very specific properties of coordinate i under row removal

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Preparation: Some linear algebra

Coja-Oghlan, Ergür, Gao, Hetterich & Rolvien considered the following concept (with minor modification):

Definition (Linear e-relations)

Let $A \in \mathbb{F}^{m \times n}$.
A set $I \subseteq [n]$ is an **e-relation** of A if there exists a row vector $y \in \mathbb{F}^{1 \times m}$ such that $\mathrm{supp}(yA) = I$. In this case, we call y a **representation** of I in A.

Atomic Linear Relations

Definition (Frozen variables)

Let $A \in \mathbb{F}^{m \times n}$.
If $I = \{i\}$ is an e-relation of A, then we call i **frozen** in A. We denote the set of all frozen variables by $\mathcal{F}(A)$.

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"$\Longleftarrow$" If $y_i = 0$ for all $y \in \ker_{\mathbb{F}}(A)$, then adding the equation $x_i = 0$ does not change the set of solutions to the homogeneous system, so it can be obtained through elementary row operations.

More kinds of frozen variables

Definition

For any matrix $A \in \mathbb{F}^{m \times n}$ and $1 \leq i \leq m \wedge n$, we say that

1. i is **firmly frozen** in A, if i is frozen in the matrix that arises from A through deletion of row i.

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2. i is **completely frozen** in A, if i is firmly frozen both in A and A^T.

Observation: Variables that are firmly frozen in A are frozen in A.

Rank and symmetric row and column removal

With the previous definitions, we have

$$\mathrm{rk}(A) - \mathrm{rk}(A\langle i; i\rangle) =$$

Rank and symmetric row and column removal

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$\Longrightarrow$ Next step: Get a handle on completely frozen and "nowhere frozen" variables.

Even more kinds of frozen variables

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We next introduce a **partition of the coordinates** into $p+3$ different types of variables.

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Definition

For any matrix $A \in \mathbb{F}^{m \times n}$ and $1 \leq i \leq m \wedge n$, we say that i is **frailly frozen** in A if i is frozen in A, but not in the matrix that arises from A through deletion of row i.

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Observation:

Proposition

Let $A \in \mathbb{F}^{m\times n}$ and $i \in [m \wedge n]$. Then

$$i \text{ is frailly frozen in } A \iff i \text{ is frailly frozen in } A^T.$$

Signature

Observation: If $\{i\}$ is either frailly frozen or completely frozen in A, then for any two representations $y = (y_1, \ldots, y_m)$, $y' = (y'_1, \ldots, y'_m)$ of $\{i\}$ in A,

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Definition (Signature)

Assume that $i \in [m \wedge n]$ is frailly frozen or completely frozen in $A \in \mathbb{F}^{m \times n}$. We call the **unique** value y_i in any representation y of i in A the signature of i in A and denote it by $\mathrm{sig}(A, i)$.

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Remark: If $\{i\}$ is completely frozen in A, then

$$\mathrm{sig}(A, i) = 0.$$

Partition of the set of variables

For a symmetric matrix $A \in \mathbb{F}^{n \times n}$, we thus have the following partition of $[n]$:

Let $i \in [n]$. Then i belongs to exactly one of the following sets:

- The set of variables that are **frailly frozen** in A **with signature** $j \in [p-1]$.
- The set of variables that are **completely frozen** in A.
- The set of variables that are **neither frozen** in A nor A^t.
- The set of variables that are **not frozen in** A**,** but **firmly frozen in** A^t.
- The set of variables that are **firmly frozen in** A, but **not frozen in** A^t.

Fixed point equations for the proportions

Now as $n \to \infty$, what are the various proportions of variables in $\boldsymbol{A}_{n,p}$?

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To characterise them, we derive equations on the change of the various proportions under the operation of symmetrically attaching a row and a column to a general matrix A:

$$A \qquad \longrightarrow \qquad A^{\boldsymbol{b}} = \begin{pmatrix} A & \boldsymbol{b}^t \\ \boldsymbol{b} & 0 \end{pmatrix}.$$

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More precisely, we derive the corresponding equations for a perturbed version of a general matrix A that eliminates most short linear relations.

Symmetric perturbation

Idea:

Apply a random perturbation to A that does not change the rank by a significant amount, but eliminates most of the short linear relations.

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$$A[3,4] = \left(\begin{array}{cccccccccc|cccc}
 & & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 1 \\
 & & & & A & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Extended perturbation

Actually, we derive equations on the change of the various proportions under the operation of symmetrically attaching a row and a column to $A[\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]$:

$$A[\boldsymbol{\theta}_1, \boldsymbol{\theta}_2] \qquad \longrightarrow \qquad A[\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]^{\boldsymbol{b}} = \begin{pmatrix} A[\boldsymbol{\theta}_1, \boldsymbol{\theta}_2] & \boldsymbol{b}^t \\ \boldsymbol{b} & 0 \end{pmatrix}.$$

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- The proportions of frailly frozen as well as "one-sided" firmly frozen variables are negligible.
- The proportion of frozen variables approximately satisfies the equation $\alpha = 1 - \phi(1 - \phi(\alpha))$.

Case 2:

- The proportion of frailly frozen variables is not negligible, but a specific function of the proportion of frozen variables.
- The proportion of completely frozen variables is not negligible, but a specific function of the proportion of frozen variables.
- The proportion of frozen variables approximately satisfies the equation

$$\alpha = 1 - \frac{1}{p}\phi(\alpha) - \frac{p-1}{p}\phi\left(\frac{p}{p-1}\left(1 - \phi\left(\alpha\right)\right) - \frac{1}{p-1}\alpha\right).$$

Lower bound on the rank

With this a priori estimate, it is possible to show that in any case, the desired lower bound holds:

Theorem

For any $d > 0$ and any prime p,

$$\liminf_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\,\mathrm{rk}_{\mathbb{F}_p}\left(\boldsymbol{A}_{n,p}\right)\right] \geq 2 - \max_{\alpha\in[0,1]} M_d(\alpha).$$

Future directions

- Extension to general entries from $\mathbb{F}_p$
- Extension to a broader class of symmetric matrices
- Rank of the adjacency matrix of the $k-$core of $\boldsymbol{G}_{n,p}$ over finite fields ($k \geq 3$)